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Microlocal analyticity of Feynman integrals

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Abstract

We give a rigorous construction of analytically regularized Feynman integrals in D-dimensional Minkowski space as meromorphic distributions in the external momenta, both in the momentum and parametric representation. We show that their pole structure is given by the usual power-counting formula and that their singular support is contained in a microlocal generalization of the $(+\alpha)$-Landau surfaces. As further applications, we give a construction of dimensionally regularized integrals in Minkowski space and prove discontinuity formula for parametric amplitudes.

Zusammenfassung

Wir geben eine rigourose Konstruktion von analytisch-regularisierten Feynman-Integralen im D-dimensionalen Minkowski-Raum als meromorphe Distributionen in den externen Impulsen, sowohl in der Impuls- als auch in der parametrischen Darstellung. Wir zeigen, dass ihre Pole durch die üblichen Power-counting Formeln gegeben sind, und dass ihr singulärer Träger in mikrolokalen Verallgemeinerungen der $(+\alpha)$-Landauflächen enthalten ist. Als weitere Anwendungen geben wir eine Konstruktion von dimensional regularisierten Integralen im Minkowski-Raum und beweisen Diskontinuitätsformeln für parametrische Amplituden.

Contents

1 Introduction

1.1 Perturbative quantum field theory

Quantum field theory is the hugely successful symbiosis of quantum mechanics and special relativity. It provides some of the most accurately verified theoretical prediction in all of physics and distills the myriad particle phenomena into an elegant and consistent framework. Yet, for a mathematician, the derivations and calculations in QFT often seem wondrous and sometimes even scandalous. Certain things like the Feynman path integral are, mathematically speaking, still indistinguishable from magic. This thesis tries to shed some light on some of the mathematical structures appearing in perturbative quantum field theory.

A main concern of perturbative QFT is the predictions of particle scattering experiments. These predictions are given by summing amplitudes of certain graphs naturally associated to the theory. For a graph G in a D-dimensional scalar theory, such an amplitude takes the form

$$I_G(p) = \int_{\mathbb{R}^{D|E_G|}} \prod_{v \in V_G} \delta^D(p_v + k_v) \prod_{j \in E_G} \Delta(k_j, m_j) \mathrm{d}^D k_j.$$

Here E_G is the set of edges of G, V_G its set of vertices and

$$\Delta(k, m) = \frac{1}{k^2 - m^2 + i\epsilon}$$

is the Feynman propagator, expressed in terms of the Minkowski metric

$$k^2 = k_0^2 - \sum_{i=1}^{D-1} k_i^2.$$

Remark 1.1.1. We have somewhat simplified the usual Feynman rules, ignoring factors of i, π and the coupling constants, since these will be irrelevant for our purposes. We also deleted the external half-edges which are usual attached to a Feynman graph and combined the corresponding external momenta into the associated vertex.

Two issues are immediately apparent when looking at the amplitude I_G. First, it is not clear that the integrand falls off sufficiently rapidly at infinity to ensure convergence. In fact this is often false and to obtain mathematically and physically meaningful results, one employs an intricate subtraction procedure called a renormalization scheme (See e.g. [Col84]). We will not consider renormalized amplitudes here, but will stop at the half-way mark and consider regularized integrals. Most of the time, we will consider analytically regularized integrals, which consist of the replacement

$$(k_j^2 - m_j^2 + i\epsilon)^{-1} \mapsto (k_j^2 - m_j^2 + i\epsilon)^{-\lambda_j}.$$

where $\lambda_j \in \mathbb{C}$ is a complex parameter. The expectation is then that a judicious choice of λ_j dampens the growth of the integrand enough to yield convergent integrals. This becomes subtle when some of the masses m_j are vanishing. Improving the convergence at infinity then worsens the singularity at $k_j = 0$ and vice versa. We will see in Chapter 7, when this balancing act is successful.

A more intricate issue is the singularity of the propagator at $k^2 = m^2$. The usual ploy to avoid the singular locus consist of giving the denominator a small imaginary part $i\epsilon$ as we did above. But what is usually meant by this expression in the QFT literature, is in fact the limit

$$\frac{1}{k_j^2 - m_j^2 + i0} = \lim_{\epsilon \to 0^+} \frac{1}{k_j^2 - m_j^2 + i\epsilon}.$$

This limit does not exist in the space of smooth functions and must be understood as a distribution. Understanding these boundary value distributions will make up a large part of this thesis.

Having now a clear view of the problems before us, it is time to look for solutions. We advocate in favour of two mathematical frameworks: Toric geometry and algebraic analysis.

1.2 Toric geometry

To understand the possible divergences describe above, we will usually try to construct explicit algebraic compactification of the integration domains. In fact, we will only need to consider rather special compactifications, which are constructed as iterated blow-ups of projective space or products of projective spaces along coordinate linear subspaces. These are examples of smooth toric varieties and we can use the powerful combinatorial machinery of toric geometry to understand their structure.

We will review the theory of toric varieties in chapter 3. Our exposition stresses the role of Cox's homogeneous coordinate ring [Cox95] and the associated quotient construction, which expresses any smooth, toric variety without torus factors as the geometric quotient

$$X_\Sigma = \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma // G_\Sigma.$$

This allows us to express many constructions on the variety X_Σ in terms of the global affine coordinates of $\mathbb{C}^{\Sigma(1)}$. In chapter 5, we will give a construction of distributional densities on the real locus $X_\Sigma(\mathbb{R})$ in terms of appropriate distributions on $\mathbb{R}^{\Sigma(1)}$, which might be of independent interest.

The varieties we are interested in actually turn out to be toric versions of the *wonderful model compactifications* of De Concini and Processi [DCP95]. We will give a convenient description of these varieties in terms of building and nested sets, based on the combinatorial results of Feichtner and Kozlov [FK04].

Remark 1.2.1. A special case of toric wonderful models associated to Feynman graphs were introduced by Bloch, Esnault and Kreimer in the seminal paper [BEK06], to study motives and periods of scaleless graphs. Their construction was later extended by Brown [Bro17] to graphs with nontrivial kinematics. The toric structure of these varieties was used in [BK08] to relate the process of renormalization to mixed Hodge structures.

Wonderful models were also used by Berghoff in [Ber15], to describe position-space amplitudes.

1.3 Algebraic and microlocal analysis

A general setting to study boundary values of holomorphic functions is the following: Let M be a real analytic manifold. A complexification of M is an embedding $M \hookrightarrow X$ into a complex manifold X which is locally isomorphic to the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$. Let $\Omega \subseteq X$ be an open subset which contains M in its closure. We then want to obtain distributions on M from appropriate holomorphic functions on Ω.

These kind of questions are intensely studied by Sato's school of algebraic analysis. Sato introduced the more general space of *hyperfunctions* $\mathcal{B}_M$, which admit a very general (but quite formal) notion of boundary values.

A key point of Sato's theory is to keep track of the direction of the boundary values by associating to every distribution or hyperfunction u on the manifold M a subset in its conormal bundle $SS(u) \subseteq T^*_M X$. Operations which are usually ill-defined for distributions, like restrictions or taking products, are then possible if their singular support $SS(u)$ is sufficiently non-characteristic. These kinds of arguments now go under the name of *microlocal analysis*.

Remark 1.3.1. Feynman integrals have been intensively studied with the techniques of algebraic and microlocal analysis in the seventies (see e.g. [SMJO76], [KK76], [KK77], [KK]). The present work can be seen as a natural continuation of their efforts with modern and hopefully more accessible methods.

An introduction to the classical theory of hyperfunctions can be found in [KKK86] and [SKK72]. We will in fact go a slightly different route and base our presentation on the moderate cohomology functor of Kashiwara and Schapira [KS96] as well as the microlocal sheaf theory developed by the same authors in [KS94]. This will allow us to give a modern and flexible account of boundary value distributions and their microlocal structure, and relate it more naturally to the classical theory of distributions developed in e.g. [Hör98].

Our exposition centers on the boundary values from *admissible* open subset $\Omega \subseteq X$, which are defined by a mild regularity condition on their boundary $\partial\Omega$. The non-characteristic conditions above can then be expressed in terms of the geometry of these open subsets using the microlocal sheaf theory of [KS94].

Example 1.3.2. Suppose $f : X \to \mathbb{C}$ is a holomorphic function, which is real-valued on M and such that 0 is not a critical value. Then $\Omega_f = \{z \in X \mid \operatorname{Im}(f(z)) > 0\}$ is an admissible subset in a neighbourhood of $f^{-1}(0)$ and

$$\frac{1}{(f(x)+i0)^{-\lambda}} := b_{\Omega_f}(f(z)^{-\lambda}),$$

extends to a globally defined distribution.

Applying this with $f(k) = k^2 - m^2$ for $m > 0$ gives the interpretation of the (analytically regularized) Feynman propagator

$$\Delta(k, m, \lambda) = \frac{1}{(k^2 - m^2 + i0)^{\lambda}},$$

we will use throughout this thesis. Note that the same argument gives a construction of the massless propagator $\Delta(k, 0, \lambda)$ as a distribution on $\mathbb{R}^D \backslash \{0\}$, but we can not immediately define $\Delta(k, 0, \lambda)$ at the singular point $k = 0$. We will construct an extension of $\Delta(k, 0, \lambda)$ to $\mathbb{R}^D$ in chapter 6.

1.4 Regularized amplitudes

Coming back to Feynman graphs, we will give a rigorous construction of the analytically regularized Feynman integral $\overline{I}_G(\lambda)$, by extending it to suitable toric compactification of the integration domain. This approach is based on the work of Sato et. al.[SMJO76], and we give an alternative proof of the following result of loc.cit.

Theorem 1.4.1. *The regularized amplitude $\overline{I}_G(\lambda)$ is a well-defined, meromorphic distribution on $V_G^{ext}(\mathbb{R}^D)$, the space of D-dimensional external momenta. Its singular support satisfies*

$$SS(\overline{I}_G(\lambda)) \subseteq \bigcup_{\substack{\gamma \subseteq G, \eta \subseteq G^0 \\ E_\gamma \cap E_\eta = \emptyset}} \mathcal{L}^+_{G \backslash \eta / \gamma},$$

where $G^0 \subseteq G$ is the subgraph consisting of all massless edges and $\mathcal{L}^+_{G\backslash\eta/\gamma}$ are microlocal versions of the classical $(+\alpha)$ Landau surfaces.

In the last chapter, we also consider Feynman integrals in the parametric representation. We will prove that the parametric integral

$$I_G^{par}(\lambda, D) = \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(p, x) + i0} \right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}} |\Omega|_{X_{\mathcal{B}}},$$

is a well-defined meromorphic distribution outside a small set $L_G^s \subseteq V_G^{ext}(\mathbb{R}^D)$ of special momenta. Here ψ_G and Φ_G are the first and second Symanzik polynomials and $X_{\mathcal{B}}$ is a convenient toric wonderful model, constructed as an iterated blow-up of P^{E_G}, e.g. the motic blow-up constructed in [Bro17].

A version of the Feynman trick then gives the following.

Theorem 1.4.2. *Outside L_G^s, the regularized amplitude can be expressed as*

$$\overline{I}_G(\lambda) = \delta^D(\sum_{a\in V_G^{ext}} p_a)(-i)^{h^1(G)(D-1)}\pi^{\frac{D}{2}h^1(G)}\frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)} I_G^{par}(\lambda, D).$$

This gives an immediate construction of dimensionally regularized amplitudes. For a fixed choice of parameters $\lambda_i^0 \in \mathbb{N}$, we can simply define the dimensionally regularized amplitude outside L_G^s as

$$I_G^{DR}(D) = \delta^D(\sum_{a\in V_G^{ext}} p_a)(-i)^{h^1(G)(D-1)}\pi^{\frac{D}{2}h^1(G)}\frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)} I_G^{par}(\lambda^0, D).$$

Remark 1.4.3. For scalar graphs, we would set $\lambda_i^0 = 1$, but other propagator structures naturally appear in the tensor reduction of higher spin theories, see e.g. [Tar96].

1.5 Parametric discontinuity formula

Discontinuity formula and dispersion relations have a long history in quantum field theory and are still an important calculational tool. We refer to [Zwi16] for a recent review. Especially Cutkosky's cutting rules [Cut60] give a physically intuitive formula for calculating discontinuities in momentum space, based on the simple replacement

$$\frac{1}{k_j^2 - m_j^2 + i0} \mapsto (-2\pi i)\Theta(k_{j0})\delta(k_j^2 - m_j^2),$$

for the edges j of the cut. Surprisingly, these rules have only recently been given a mathematically rigorous footing by Bloch and Kreimer [BK15], using the geometric methods of Pham [Pha].

We will not consider Cutkosky rules in this thesis, but will instead supplement the above formula by proving a (apparently new) discontinuity formula in the parametric representation. We will restrict to massive 2-point graphs, i.e. graphs with 2 external vertices. Then the parametric amplitude $I^{par}(\lambda, D, p) = I^{par}(\lambda, D, s)$ can be expressed in terms of the channel variable $s = p_a^2$, where p_a is one of the two external vertices. It is easy to show that $I^{par}(\lambda, D, s)$ is the boundary value

$$I^{par}(\lambda, D, s) = \tilde{I}(\lambda, D, s + i0),$$

where $\tilde{I}(\lambda, D, s)$ is a multivalued function in $s \in \mathbb{C}$, which has a single-valued branch on $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$. We define the discontinuity of $I_G^{par}(\lambda, D, s)$ as

$$\operatorname{Disc} I_G^{par} = \tilde{I}(\lambda, D, s + i0) - \tilde{I}(\lambda, D, s - i0).$$

The following discontinuity formula then follows easy from the formalism developed in this thesis.

Theorem 1.5.1. *The discontinuity*

$$\operatorname{Disc} I_G^{par} = \tilde{I}(\lambda, D, s + i0) - \tilde{I}(\lambda, D, s - i0).$$

can be expressed as the integral

$$\operatorname{Disc} I_G^{par} = (1 - e^{-2\pi i \omega_G}) \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\chi_+^{-\omega_G}(\Phi_G(s,x))\right) \psi_G(x)^{\omega_G - \frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}.$$

For overall convergent, scalar graphs in $D^0 \in \mathbb{N}$ dimensions, the above formula simplifies to

$$\operatorname{Disc} I_G^{par} = -2\pi i \int_{X_{\mathcal{B}}(\mathbb{R})} \left(\delta^{(\omega_G - 1)}(\Phi_G(s,x))\right) \psi_G(x)^{\omega_G - \frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}.$$

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2 Microlocal sheaf theory

In this chapter, we briefly review the theory of sheaves on topological spaces and the microlocal techniques developed in [KS94], which will serve as the basis of our discussion of microlocal analyticity. We refer to op.cit. for further details and most proofs.

2.1 Sheaves on locally compact spaces

Suppose X is a suitable nice topological space. For now, we assume that X is a locally compact Hausdorff space. Let

$$Op_X = \{U \subseteq X \mid U \text{ open}\}$$

be the partially ordered set of open subsets of X. We can regard Op_X as a category in the usual way.

Definition 2.1.1. A presheaf on X is a contravariant functor $F : Op_X^{op} \to \mathrm{Set}$.

Hence a presheaf is given by a set $F(U)$ for every open set $U \subseteq X$ and a restriction map

$$F(U) \to F(V), \quad s \mapsto s|_V$$

for every inclusion $V \hookrightarrow U$. If $U \in Op_X$ and $U = \bigcup_{i\in\mathcal{I}} U_i$ is an open cover, we have the diagram

$$F(U) \longrightarrow \prod_{i\in\mathcal{I}} F(U_i) \rightrightarrows \prod_{i,j\in\mathcal{I}} F(U_i \cap U_j).$$

The left arrow is the product of the restriction maps $s \mapsto s|_{U_i}$ The first arrow of the pair on the right is the product over all $i \in \mathcal{I}$ of the maps $F(U_i) \to \prod_{j\in\mathcal{I}} F(U_i \cap U_j)$, while the second is the product over all $j \in \mathcal{I}$ of $\prod_{i\in\mathcal{I}} F(U_i) \to \prod_{i\in\mathcal{I}} F(U_i \cap U_j)$.

Definition 2.1.2. A presheaf F is a sheaf if, for every open $U \in Op_X$ and open cover $U = \bigcup_{i\in\mathcal{I}} U_i$, the sequence

$$F(U) \longrightarrow \prod_{i\in\mathcal{I}} F(U_i) \rightrightarrows \prod_{i,j\in\mathcal{I}} F(U_i \cap U_j),$$

is an equalizer diagram.

More prosaically, a presheaf F is a sheaf if for $U = \bigcup_{i\in\mathcal{I}} U_i$ as above:

1. If $s, t \in F(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in \mathcal{I}$, then $s = t$.

2. If there are elements $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in \mathcal{I}$, then there is $s \in F(U)$ with $s|_{U_i} = s_i$.

Example 2.1.3. For $U \in Op_X$, let $C^0(U)$ be the set of continuous functions on U. Restriction of functions gives the presheaf morphisms $C^0_X(U) \to C^0(V)$ and it is easy to see that C^0 is a sheaf. If X is a smooth (resp. analytic) manifold, then we can similarly define the sheaves C^∞_X and $\mathcal{A}_X$ of smooth (resp. analytic) functions.

On the other hand, the presheaf $U \mapsto L^1_X(U)$ of absolutely integral functions is not a sheaf, since a locally integrable function need not be integrable.

Let $\mathrm{PSh}(X) = \mathrm{Func}(Op_X^{op}, \mathrm{Set})$ be the category of presheaves on X and $\mathrm{Sh}(X)$ the full subcategory of sheaves.

Proposition 2.1.4 ([KS94, Prop. 2.2.3])**.** *The inclusion $i : \mathrm{Sh}(X) \to \mathrm{PSh}(X)$ has a natural left-adjoint, the sheafification functor $a : \mathrm{PSh}(X) \to \mathrm{Sh}(X)$. Hence for every presheaf $F \in \mathrm{PSh}(X)$, there is a sheaf $a(P)$, such that*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(a(F), G) \cong \mathrm{Hom}_{\mathrm{PSh}(X)}(F, i(G))$$

for every sheaf $G \in \mathrm{Sh}(X)$.

Example 2.1.5. Let A be a set and define the presheaf A^{ps}_X by $A_X(U) = A$. The sheafification $A_X := a(A^{ps}_X)$ is the sheaf of locally constant functions with values in A. A sheaf F is called constant if there is a set A, such that $F \cong A_X$.

Example 2.1.6. The sheafification $a(L^1_X)$ is the sheaf of locally integrable functions $L^{1,loc}_X$.

Now suppose A is a ring. For simplicity, we will either assume that $A = \mathbb{Z}$ or A is field of zero characteristic. Let Mod_A be the abelian category of A-modules. We denote the category of sheaves with values in Mod_A by $\mathrm{Mod}(A_X)$, i.e. $\mathrm{Mod}(A_X)$ is the full subcategory of functors $Op_X^{op} \to A$ satisfying the sheaf conditions.

This is naturally an abelian category: If $\phi : F \to G$ is a morphism in $\mathrm{Mod}(A_X)$, then

$$\ker \phi(U) = \ker\left(\phi(U) : F(U) \to G(U)\right),$$

and $\operatorname{coker} \phi$ is the sheaf associated to the presheaf $U \mapsto G(U)/\phi(F(U))$.

We also have internal morphism and tensor product functors

$$\begin{aligned} \mathcal{H}om_{A_X}(-,-) &: \mathrm{Mod}(A_X)^{op} \times \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_X), \\ - \otimes_{A_X} - &: \mathrm{Mod}(A_X) \times \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_X). \end{aligned}$$

The first is the sheaf $U \mapsto \mathcal{H}om_A(F(U), G(U))$ and the second is the sheaf associated to the presheaf $U \mapsto F(U) \otimes_A G(U)$. They satisfy the usual adjunction, i.e.

$$\begin{aligned} \mathcal{H}om_{A_X}(F \otimes_{A_X} G, H) &\cong \mathcal{H}om_{A_X}(F, \mathcal{H}om_{A_X}(G, H)), \\ \mathrm{Hom}_{\mathrm{Mod}(A_X)}(F \otimes_{A_X} G, H) &\cong \mathrm{Hom}_{\mathrm{Mod}(A_X)}(F, \mathcal{H}om_{A_X}(G, H)). \end{aligned}$$

Let
$$\Gamma : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A), \quad F \mapsto F(X)$$
be the global sections functor. We can express the morphism spaces in $\mathrm{Mod}(A_X)$ as
$$\mathrm{Hom}_{A_X}(F, G) = \Gamma(X, \mathcal{H}om_{A_X}(F, G)).$$
In the sequel, we will omit the subscript A_X if it is clear from the context.

Remark 2.1.7. Note that $F \in \mathrm{Mod}(A_X)$ if and only if it is a sheaf of abelian groups, i.e. $F \in \mathrm{Mod}(\mathbb{Z}_X)$, and there is a natural map
$$A_X \otimes_{\mathbb{Z}_X} F \to F.$$
More generally, let $\mathcal{R}_X$ be a sheaf of rings, i.e. a sheaf with values in the category of rings. An $\mathcal{R}_X$-module is a sheaf of abelian groups $M \in \mathrm{Mod}(\mathbb{Z}_X)$, with a natural map $\mathcal{R}_X \otimes_{\mathbb{Z}_X} M \to M$ satisfying the usual associativity and unitality constrains. If $\mathcal{R}_X$ is abelian, we again have internal tensor product and morphism functors
$$- \otimes_{\mathcal{R}_X} - : \mathrm{Mod}(\mathcal{R}_X) \times \mathrm{Mod}(\mathcal{R}_X) \to \mathrm{Mod}(\mathcal{R}_X),$$
$$\mathcal{H}om_{\mathcal{R}_X}(-,-) : \mathrm{Mod}(\mathcal{R}_X)^{op} \times \mathrm{Mod}(\mathcal{R}_X) \to \mathrm{Mod}(\mathcal{R}_X),$$
where $\mathrm{Mod}(\mathcal{R}_X)$ denotes the category of $\mathcal{R}_X$-modules. In the later sections, we will consider the sheaves of rings $\mathcal{O}_X$ and $\mathcal{D}_X$ of holomorphic functions and holomorphic differential operators on a complex manifold X.

For a continuous map $f : X \to Y$, we have natural pullback and pushforward functors
$$f_* : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_Y) : f^{-1},$$
which are adjoint to each other. f_* is given by
$$f_*F(V) = F(f^{-1}(V))$$
and f^{-1} is the sheaf associated to the presheaf
$$V \mapsto \varinjlim_{f(V) \subseteq U} F(U),$$
where the colimit ranges over all open neighbourhoods U of $f(V)$. If $f : \{x\} \hookrightarrow X$ is the inclusion of a point, we obtain the stalk $F_x = \varinjlim_{x \in U} F(U)$ of F at x. The support $\mathrm{supp}(s)$ of a section $s \in F(U)$ is the closure of the set $\{x \in X | s_x = 0\}$, where s_x is the image of s under the restriction map $F(U) \to F_x$. In other words, the complement $V = U \setminus \mathrm{supp}(s)$ is the largest open set, such that $s|_V = 0$.

A useful variation of f_* is the proper pushforward
$$\begin{aligned} &f_! : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_Y) \\ &\qquad f_!F(V) := \{s \in F(f^{-1}(V)) \mid f : \mathrm{supp}(s) \to V \text{ is proper}\}. \end{aligned}$$

To a locally closed subset $Z \subseteq X$ we can associate two new functors as follows. Let $i : Z \hookrightarrow X$ be the inclusion. We set

$$(-)_Z : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_X), \quad F_Z = i_! i^{-1} F,$$
$$\Gamma_Z : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_X), \quad \Gamma_Z(F) = \mathcal{H}om(A_Z, F).$$

We will also write

$$\Gamma_Z(X, F) := \Gamma(X, \Gamma_Z(F)), \quad \Gamma(Z, F) := \Gamma(X, i^* F),$$

for the corresponding global sections.

Example 2.1.8. Suppose Z is closed. Then i is proper and $F_Z = i_* i^{-1} F$ is given by the sections defined in a neighbourhood of Z. The sheaf $\Gamma_Z(F)$ consists of sections with support in Z.

Example 2.1.9. Let $j : U \to X$ be the inclusion of an open subset. Then $F_U = j_! j^{-1}(F)$ consists of sections which have support contained in U. We also have an isomorphism $\Gamma_U(F) = j_* j^{-1} F$, i.e. $\Gamma_U F$ is the sheaf whose sections over $V \subseteq X$ are given by $\Gamma_U(F)(V) = F(V \cap U)$.

A key point of the formalism developed in [KS94] is working consistently in the derived category. We refer to ([KS94], [Wei94], [KS06], [GM03]) for background on triangulated and derived categories.

Let $D^*(A_X)$ (for $* \in \{+, -, b\}$) be the (bounded above, bounded below, bounded) derived category of A_X-modules. The category $\mathrm{Mod}(A_X)$ has enough injectives so that we can derive all left exact functors. In particular we get functors

$$\begin{aligned} R\Gamma : D^+(A_X) &\longrightarrow D^+(A) \\ R\Gamma_c : D^+(A_X) &\longrightarrow D^+(A) \\ R\Gamma_Z : D^+(A_X) &\longrightarrow D^+(A_X) \\ R\mathcal{H}om(\cdot, \cdot) : D^-(A_X)^{op} \times D^+(A_X) &\longrightarrow D^+(A_X) \\ R\,\mathrm{Hom}(\cdot, \cdot) : D^-(A_X)^{op} \times D^+(A_X) &\longrightarrow D^+(A) \\ Rf_* : D^+(A_X) &\longrightarrow D^+(A_Y) \\ Rf_! : D^+(A_X) &\longrightarrow D^+(A_Y) \end{aligned}$$

The functors f^{-1} and $(\cdot)_Z$ are exact, so they extend immediately to functors

$$\begin{aligned} f^{-1} : D^*(A_Y) &\longrightarrow D^*(A_X) \\ (\cdot)_Z : D^*(A_X) &\longrightarrow D^*(A_X). \end{aligned}$$

Our hypothesis on the coefficient ring A also imply that every A_X-module has a finite flat resolution. Therefore we also have a functor

$$- \otimes^L_{A_X} - : D^*(A_X) \times D^*(A_X) \to D^*(A_X).$$

For the spaces we are considering, these functors usually have finite cohomological dimension. Hence we can safely work in the bounded derived category $D^b(A_X)$, which we will do from now own without further mention.

Example 2.1.10. If $Z \subseteq X$ is a closed set, $U = X \backslash Z$ and F an A_X-module, then there is an exact sequence

$$0 \longrightarrow F_U \longrightarrow F \longrightarrow F_Z \longrightarrow 0.$$

For $F = A_X$ and $G \in D^b(A_X)$, applying $R\mathcal{H}om(\cdot, G)$ gives the distinguished triangle

$$R\Gamma_Z G \longrightarrow G \longrightarrow R\Gamma_U G \xrightarrow{+1} .$$

Example 2.1.11. In the situation above, suppose X and thus Z are compact. Applying $R\Gamma_c$ to the above exact sequence gives the triangle

$$R\Gamma_c(U, A) \longrightarrow R\Gamma(X, A) \longrightarrow R\Gamma(Z, A) \xrightarrow{+1} .$$

Hence we can identify $R\Gamma_c(U, A_X)[1]$ with the cone of $R\Gamma(X, A_X) \to R\Gamma(Z, A_X)$ and obtain isomorphisms

$$H^k_c(U, A) \cong H^k(X, Z, A).$$

The later can be identified with relative singular cohomology groups for sufficiently nice subsets U and Z.

We can compute derived functors by using appropriate acyclic resolutions.

Definition 2.1.12. An A_X-module F is

- *flabby*, if for all $U \subseteq X$ open, the natural map $\Gamma(X, F) \to \Gamma(U, F)$ is surjective.
- *c-soft*, if for all $K \subseteq X$ compact, the natural map $\Gamma(X, F) \to \Gamma(K, F)$ is surjective.

These properties are related as follows:

$$\text{injective} \Rightarrow \text{flabby} \Rightarrow \text{c-soft}$$

Example 2.1.13. The sheaf C^∞_X on a manifold X is c-soft: Let $\phi \in \Gamma(K, C^\infty_X)$, i.e. $\phi \in \Gamma(U, C^\infty_X)$ for some neighbourhood U of K. By using a partition of unity, we can construct a bump function $\rho \in \Gamma(X, C^\infty_X)$ with $\rho|_{\tilde{U}} = 1$ for a smaller neighbourhood $\tilde{U} \subseteq U$ of K and such that $\operatorname{supp} \phi \subseteq U$. Then $\rho\phi \in \Gamma(X, C^\infty_X)$ is global section of C^∞_X, whose image in $\Gamma(K, C^\infty_X)$ agrees with ϕ. This shows that C^∞_X is c-soft.

More generally, every *fine* sheaf, i.e. a sheaf with appropriate partitions of unity is c-soft.

Example 2.1.14. We will later see that $\mathcal{B}_M$, the sheaf of hyperfunctions on a real analytic manifold M, is flabby.

Proposition 2.1.15 ([KS94, Cor. 2.4.8, Prop. 2.5.8 and Prop. 2.5.10])**.** *Let $Z \subseteq X$ be a locally closed set and $f : X \to Y$ a continuous map.*

1. *Flabby sheaves are acyclic for Γ_Z.*

2. c-soft sheaves are acyclic for $\Gamma(X, \cdot), \Gamma_c(X, \cdot)$ *and* $f_!$.

The usual adjunctions between sheaves extend naturally to the derived category. For instance, we have natural isomorphisms

$$R\operatorname{Hom}(F; G) \cong R\Gamma(X, R\mathcal{H}om(F, G))$$
$$R\operatorname{Hom}(F \otimes^L G, H) \cong R\operatorname{Hom}(F, R\mathcal{H}om(G, H)).$$

We will freely use these and similar identifications and refer to [KS94, Section 2.6] for a detailed discussion.

The following properties of the push-forward and inverse image functors will be especially important in the sequel.

Proposition 2.1.16 ([KS94, Prop 2.6.4 and Prop. 2.6.6])**.** *Let* $f : Y \to X$ *be a continuous map,* $F \in D^b(A_Y)$ *and* $G \in D^b(A_X)$.

1. *There is a natural adjunction*

$$\operatorname{Hom}(f^{-1}G, F) \cong \operatorname{Hom}(G, Rf_*F)$$

induced by an isomorphism

$$R\mathcal{H}om(G, Rf_*F) \cong Rf_*R\mathcal{H}om(f^{-1}G, F).$$

2. *There is a natural morphism*

$$Rf_*(F \otimes^L f^{-1}G) \to Rf_*F \otimes^L G$$

inducing an isomorphism

$$Rf_!(F \otimes^L f^{-1}G) \to Rf_!F \otimes^L G.$$

Proposition 2.1.17 ([KS94, Prop. 2.6.7])**.** *Suppose*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

is a cartesian square of continuous maps, i.e. $Y' = X \times_X X'$. *Then there is a natural isomorphism of functors*

$$g^{-1} \circ Rf_! \cong Rf'_! \circ g'^{-1}.$$

Example 2.1.18. Let $i : S \hookrightarrow X$ be the inclusion of a locally closed subset and $f : Y \to X$ a continuous map. Let $i' : f^{-1}(S) \hookrightarrow Y$ be the pullback of i under f and $f_S = f\big|_{f^{-1}(S)}$ the restriction. Then the above proposition gives for $F \in \operatorname{Mod}(A_X)$:

$$f^{-1}(F_S) = f^{-1}i_!i^{-1}F \cong i'_!f_S^{-1}i^{-1}\mathbb{C}_S \cong i'_!i'^{-1}(f^{-1}F) \cong (f^{-1}F)_{f^{-1}(S)}.$$

2.2 Verdier duality

Let $f : Y \to X$ be continuous. We have seen that f_* and f^{-1} are naturally adjoint functors, inducing the corresponding adjoint functors Rf_* and f^{-1} in the derived categories. The proper push-forward $f_!$ generally does not have a (right or left) adjoint in the category of sheaves, but there is a right adjoint in the derived category.

Theorem 2.2.1 ([KS94, Section 3.1]). *Suppose $Rf_! : D^+(A_Y) \to D^+(A_X)$ has finite cohomological dimension, i.e. there is $r \geq 0$ such that $R^j f_! G = 0$ for $j > r$ and $G \in D^+(A_Y)$. Then there is a functor*

$$f^! : D^+(A_X) \to D^+(A_Y)$$

with the following properties:

1. *There are natural isomorphism for $F \in D^b(A_X)$ and $G \in D^b(A_Y)$:*

   $$R\operatorname{Hom}(Rf_!G, F) \cong R\operatorname{Hom}(G, f^!F)$$
   $$R\mathcal{H}\mathrm{om}(Rf_!G, F) \cong Rf_* R\mathcal{H}\mathrm{om}(G, f^!F)$$

   In particular we have the natural adjunction

   $$\operatorname{Hom}(Rf_!G, F) \cong \operatorname{Hom}(G, f^!F).$$

2. *If $g : X \to Z$ is another map of locally compact spaces such that $g_!$ has finite cohomological dimension, then there is a natural isomorphism*

   $$(g \circ f)^! \cong f^! \circ g^!.$$

Definition 2.2.2. For a map $f : Y \to X$ as above, we call

$$\omega_{Y/X} := f^! A_X$$

the relative dualizing complex. For $X = \{pt\}$ we also write $\omega_Y = \omega_{Y/\{pt\}}$.

Theorem 2.2.3 ([KS94, Prop. 3.1.9-13]). *Let $F, F_1, F_2 \in D^b(A_X)$ and $f : Y \to X$ as above.*

1. *There is a natural map*

   $$f^{-1}F \otimes \omega_{Y/X} \longrightarrow f^!F.$$

2. *There is a natural isomorphism*

   $$f^! R\mathcal{H}om(F_1, F_2) \cong R\mathcal{H}om(f^{-1}F_1, f^!F_2).$$

3. *Suppose $f : Y \to X$ is the inclusion of locally closed subset $Y \subseteq X$. Then*

   $$f^!F = f^{-1}R\Gamma_Y(F).$$

4. Suppose f fits in a cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

Then there is a natural isomorphism of functors

$$f^! \circ Rg_* \cong Rg'_* \circ f'^!.$$

Now let us specialize the above to the case of manifolds.

Proposition 2.2.4 ([KS94, Prop. 3.3.2]). *Suppose Y and X are manifolds and $f : Y \to X$ is a submersion of relative dimension $d = \dim Y - \dim X$.*

i) The complex $\omega_{Y/X}$ is concentrated in degree $-d$, i.e. $\omega_{Y/X} \simeq H^{-d}(\omega_{Y/X})[d]$ and is locally isomorphic to $A_Y[d]$.

ii) The natural morphism for $F \in D^b(A_X)$,

$$\omega_{Y/X} \otimes f^{-1}(F) \longrightarrow f^!(F)$$

is an isomorphism.

Definition 2.2.5. For a submersion $f : Y \to X$ of relative dimension d, we let

$$or_{Y/X} = H^{-d}(\omega_{Y/X})$$

be the relative orientation sheaf. If $X = \{pt\}$, we call $or_Y = or_{Y/\{pt\}}$ the orientation sheaf.

Proposition 2.2.6. *Let X be a manifold of dimension n.*

1. The orientation sheaf or_X is the sheaf associated to the presheaf

$$V \mapsto \mathrm{Hom}(H^n_c(V, A_X), A).$$

2. There is a natural isomorphism

$$or_X \otimes or_X \cong A_X.$$

An orientation of X defines a global isomorphism

$$or_X \cong A_X.$$

3. Let Y be another manifold and $f : Y \to X$ be a continuous map of relative dimension d. Then there is a natural isomorphism

$$\omega_{Y/X} \cong (f^{-1}\omega_X)^{-1} \otimes \omega_Y \cong (f^{-1}or_X) \otimes or_Y[d],$$

where we set $F^{-1} = R\mathcal{H}om(F, A_Y)$ for $F \in D^b(A_Y)$.

We sketch the proof to give some idea of how the above definition relates to the usual notion of orientation.

Proof sketch.

1. For $U \cong \mathbb{R}^n$ an open subset in X and $a_X : X \to \{pt\}$ we have

$$\begin{aligned} R^0\Gamma(U, \omega_X[-n]) &\cong R^0 \operatorname{Hom}(A_U, \omega_X) \\ &\cong R^0 \operatorname{Hom}(A_U, a_X^! A) \\ &\cong R^0 \operatorname{Hom}(R\Gamma_c(U, A_X), A) \\ &\cong \operatorname{Hom}(H_c^n(U, A_X), A)[n]. \end{aligned}$$

   In the last line we have used that $R\Gamma_c(U, A_X) \cong A[-n]$, which can be seen as follows: Let $U \cong \mathbb{R}^n \hookrightarrow S^n$ be the one-point compactification. Example 2.1.11 then gives the isomorphism

$$H_c^k(U, A_X) \cong \tilde{H}^k(S^n, A),$$

   where the left hand side is the reduced cohomology of S^n, which is only nonzero in degree n. In particular, an orientation of U in the conventional sense picks out an isomorphism

$$H_c^n(U, A_X) \cong H^n(S^n, A) \cong A,$$

   which is defined over $\mathbb{Z} \subseteq A$.

2. Let $X = \bigcup U_i$ be an open cover with $U_i \cong \mathbb{R}^n$. We have seen above that choosing orientations on the U_i gives isomorphisms $or_X\big|_{U_i} \cong A_X\big|_{U_i}$.
   On the overlaps $U_i \cap U_j$ we have two different identification which might differ by an automorphism of $A_X\big|_{U_i \cap U_j}$. It is shown in [KS94, Lemma 3.3.7], that this automorphism is the identity if the orientations are compatible, i.e. if the U_i are part of an oriented atlas. Hence if X is oriented, these isomorphisms glue to a global identification $or_X \cong A_X$. The induced maps on the tensor product

$$or_X\big|_{U_i} \otimes or_X\big|_{U_i} \cong A_X\big|_{U_i} \otimes A_X\big|_{U_i} = A_X\big|_{U_i}$$

   always glue, since we can assume they are defined over $\mathbb{Z}$ and the sign differences cancel.

3. Since ω_Y is locally isomorphic to $A_Y[\dim Y]$ and

$$R\mathcal{H}om(\omega_X, \omega_X) \cong or_X \otimes or_X \cong A_X,$$

   we obtain from Thm. 2.2.3, that

$$\begin{aligned} \omega_{Y/X} = f^! R\mathcal{H}om(\omega_X, \omega_X) &\cong R\mathcal{H}om(f^{-1}\omega_X, \omega_Y) \\ &\cong R\mathcal{H}om(f^{-1}\omega_X, A_Y) \otimes \omega_Y = (f^{-1}\omega_X)^{-1} \otimes \omega_Y \\ &\cong (f^{-1}or_X)^{-1} \otimes or_Y[d]. \end{aligned}$$

□

Remark 2.2.7. To ease notation, we will often abbreviate $f^{-1}\omega_X$ (resp. $f^{-1}or_X$) to ω_X (resp. or_X) if the map f is clear from context. The previous proposition also makes it natural to set

$$or_{Y/X} = H^{-d}(\omega_{Y/X}) \cong (f^{-1}or_X)^{-1} \otimes or_Y,$$

even if f is not a submersion.

2.3 Distributions and resolutions

Let us use our result on the orientation sheaf to introduce some more examples of sheaves, which will play a major role in the sequel. We refer to [KS94, Section 2.9] for more detailed discussion.

Suppose X is a smooth manifold of dimension n. For $0 \le k \le n$ we denote by $C_X^{\infty,k}$ the sheaf of k-forms on X and by

$$C_X^{\infty,\vee} = C_X^{\infty,n} \otimes or_X$$

the sheaf of smooth densities.

For $U \in Op_X$ we let $Db_X(U)$ be the space of distributions on U, the topological dual of the space of $\Gamma_c(U, C_X^{\infty,\vee})$ smooth densities on U with compact support.

If $V \subseteq U$ is another open subset and $\phi \in C_X^\infty(V)$ has compact support on V, then we can extend it by zero to a compactly supported smooth function $\phi_U \in C_X^\infty(U)$. The restriction map $Db_X(U) \to Db_X(V)$ is then defined by duality:

$$\langle u|_V, u\rangle = \langle u, \phi_U\rangle,$$

where $u \in Db_X(U)$ and $\langle -,-\rangle$ denotes the pairing between distributions and smooth densities. Thus we obtain a presheaf Db_X.

Proposition 2.3.1. *The presheaf Db_X is a sheaf.*

Proof. Let $U = \bigcup_{i\in\mathcal{I}} U_i$ be an open cover of U and $u_i \in Db_X(u_i)$ distributions, with $u_i|_{U_i\cap U_j} = u_j|_{U_i\cap U_j}$. Let $(\rho_i)_{i\in\mathcal{I}}$ be a partition of unity subordinate to $(U_i)_{i\in\mathcal{I}}$. Then we can define $u \in Db_X(U)$ by

$$\langle u, \phi\rangle = \sum_{i\in\mathcal{I}} \langle u_i, \rho_i\phi\rangle.$$

It is straightforward to check that this gives the unique section of $Db_X(U)$ with $u|_{U_i} = u_i$. ☐

More generally, we define the space of degree k currents $Db_X^k(U)$ on $U \subseteq X$ as the topological dual of $\Gamma_c(U, C_X^{\infty,n-k} \otimes or_X)$ and the space of distributional densities $Db_X^\vee(U)$ as the dual of $\Gamma_c(U, C_X^\infty)$. These give sheaves $Db_X^k, Db_X^\vee$ as above.

Example 2.3.2. For $u \in C_X^{\infty,k}(U)$ we have a natural pairing

$$\langle u, \phi\rangle = \int_X u \wedge \phi,$$

for $\phi \in \Gamma_c(U, C_X^{\infty,n-k} \otimes or_X)$. This gives an inclusion $C_X^{\infty,k} \to Db_X^k$.

Example 2.3.3. Suppose X is oriented and $Y \subseteq X$ is an oriented submanifold of codimension k. Then the pairing

$$\langle \chi_Y, \phi \rangle = \int_Y \phi,$$

for $\phi \in \Gamma_c(X, C_X^{\infty, n-k})$ defines a current $\chi_Y \in Db_X^k(X)$.

We have a natural resolution $\mathbb{C}_X \cong (C_X^{\infty,\bullet}, d)$, where the complex on the left hand side is the de Rham complex of smooth differential forms. We can extend this to a resolution $C_X \cong (Db_X^\bullet, d)$.

Proposition 2.3.4. *There are natural isomorphism*

$$R\Gamma(X, \mathbb{C}) \cong (\Gamma(X, C_X^{\infty,\bullet}), d) \cong (\Gamma(X, Db_X^\bullet), d)$$
$$R\Gamma_c(X, \mathbb{C}) \cong (\Gamma_c(X, C_X^{\infty,\bullet}), d) \cong (\Gamma_c(X, Db_X^\bullet), d)$$

In particular we have isomorphisms $H^k(X, \mathbb{C}_X) \cong H^k_{dR}(X, \mathbb{C})$ where $H^k_{dR}(X, \mathbb{C})$ are the de Rahm-cohomology groups.

Proof. The sheaves $C_X^{\infty,\bullet}$ and $Db_X^\bullet$ are c-soft so the assertion follows from Prop. 2.1.15. □

Let $f : Y \to X$ be morphism of manifolds. We can define a sheaf morphism

$$f_! Db_Y^\vee \to Db_X^\vee$$

as follows: For $u \in f_! Db_Y^\vee(U)$ and $\phi \in \Gamma_c(U, C_X^\infty)$, the set $\mathrm{supp}(u) \cap f^{-1}(\mathrm{supp}(u))$ is compact. Then the pairing

$$\langle f_! u, \phi \rangle = \langle u, f^* \phi \rangle$$

is well-defined and gives the element $f_! u \in Db_X^\vee(U)$.

Now suppose X is a complex manifolds with $\dim_{\mathbb{C}} X = n$. On X we have the sheaves $\mathcal{O}_X$ and Ω_X^k of holomorphic functions and k-forms. The Dolbeault resolution gives a quasi-isomorphism

$$\Omega_X^k \cong (C_X^{\infty,(k,\bullet)}, \overline{\partial}),$$

where $C_X^{\infty,(k,l)}$ is the sheaf of smooth differential forms of bidegree (k, l) and

$$\overline{\partial} : C_X^{\infty,(k,l)} \to C_X^{\infty,(k,l+1)},$$

is the Dolbeault differential. Using differential forms with distributional sections gives the alternative resolution

$$\Omega_X^k \cong (Db_X^{(k,\bullet)}, \overline{\partial}),$$

where $Db_X^{(k,l)}(U)$ is the topological dual of $\Gamma_c(U, C_X^{\infty,(n-k,n-l)})$.

Proposition 2.3.5. *Let $f : X \to X$ be a morphism of complex manifolds. There is a natural morphism*

$$Rf_!\Omega_Y[\dim_{\mathbb{C}} Y] \to \Omega_X[\dim_{\mathbb{C}} X].$$

Proof. Set $m = \dim_{\mathbb{C}} Y$ and $n = \dim_{\mathbb{C}} X$. We have the isomorphism

$$Rf_!\Omega_Y \cong f_! Db_Y^{m,\bullet}.$$

As above, we have a sheaf morphism

$$f_! : f_! D_Y^{m,q} \to D_X^{n,q+m-n}$$

The Dolbeault differential commutes with the pullback f^* (and then also with $f_!$) since f is holomorphic. Shifting degrees gives the map

$$Rf_!\Omega_Y[m] \cong f_! D_Y^{m,\bullet-m} \to D_X^{n,\bullet-n} \cong \Omega_X[n].$$

□

2.4 The Fourier-Sato transform

Suppose $\tau : E \to X$ is a vector bundle of rank n over a manifold X and $\pi : E^* \to X$ its dual bundle. We will identify X with the zero section of the respective bundles when appropriate. There is a natural action of $\mathbb{R}^+ = (0, \infty)$ on the fibres of E and E^*.

Definition 2.4.1. A sheaf $F \in \mathrm{Mod}(A_E)$ is called *conic* if it is constant along any $\mathbb{R}^+$-orbit. We let $D^b_{\mathbb{R}^+}(A_E) \subseteq D^b(A_E)$ be the full triangulated subcategory consisting of complexes with conic cohomology sheaves.

Set

$$P = \{(x, y) \in E \times_X E^* \mid \langle x, y\rangle \geq 0\}$$
$$P' = \{(x, y) \in E \times_X E^* \mid \langle x, y\rangle \leq 0\}$$

and let p_1, p_2 be the natural projections in the following diagram:

$$\begin{tikzcd}
& E \times_X E^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
E \arrow[dr, "\tau"'] & & E^* \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

For $F \in D^+_{\mathbb{R}^+}(A_E)$ and $G \in D^+_{\mathbb{R}^+}(A_{E^*})$, let us define the four functors

$$\tilde{\Phi}_{P'}(F) = Rp_{2!}(p_1^{-1}F)_{P'}$$
$$\tilde{\Psi}_P(F) = Rp_{2*}R\Gamma_P(p_1^{-1}F)$$
$$\tilde{\Phi}_P(G) = Rp_{1!}(p_2^!F)_P$$
$$\tilde{\Psi}_{P'}(G) = Rp_{1*}R\Gamma_{P'}(p_2^!F)$$

Proposition 2.4.2 ([KS94, Prop. 3.6.2, 3.7.4 and 3.7.7])**.** *The above definition give well-defined functors*

$$\begin{aligned}\tilde{\Phi}_{P'}, \tilde{\Psi}_P : & D^+_{\mathbb{R}^+}(A_E) \to D^+_{\mathbb{R}^+}(A_{E^*}) \\ \tilde{\Phi}_P, \tilde{\Psi}_{P'} : & D^+_{\mathbb{R}^+}(A_{E^*}) \to D^+_{\mathbb{R}^+}(A_E),\end{aligned}$$

such that the pairs $(\tilde{\Phi}_P, \tilde{\Psi}_P)$ *and* $(\tilde{\Phi}_{P'}, \tilde{\Psi}_{P'})$ *are adjoint functors and there are natural isomorphisms* $\tilde{\Phi}_{P'} \cong \tilde{\Psi}_P$ *and* $\tilde{\Phi}_P \cong \tilde{\Psi}_{P'}$*.*

Definition 2.4.3. Let $F \in D^+_{\mathbb{R}^+}(E)$ and $G \in D^+_{\mathbb{R}^+}(E^*)$.

i) The Fourier-Sato transform of F is

$$\widehat{F} = \tilde{\Phi}_{P'}(F) \cong \tilde{\Psi}_P(F) \in D^+_{\mathbb{R}^+}(E^*).$$

ii) The inverse Fourier-Sato transform is

$$\check{G} = \tilde{\Phi}_P(G) \cong \tilde{\Psi}_{P'}(G) \in D^+_{\mathbb{R}^+}(E).$$

Proposition 2.4.4 ([KS94, Thm. 3.7.9])**.** *Let* $F, F \in D^+_{\mathbb{R}^+}(E)$ *and* $G, G' \in D^+_{\mathbb{R}^+}(E^*)$*. There are natural isomorphisms*

$$\begin{aligned}R\operatorname{Hom}(F, F') &\cong R\operatorname{Hom}(\widehat{F}, \widehat{F}') \\ R\operatorname{Hom}(G, G') &\cong R\operatorname{Hom}(\check{G}, \check{G}')\end{aligned}$$

Hence the functors

$$\widecheck{(\cdot)} : D^+_{\mathbb{R}^+}(E) \to D^+_{\mathbb{R}^+}(E^*) \quad \text{and} \quad \widehat{(\cdot)} : D^+_{\mathbb{R}^+}(E^*) \to D^+_{\mathbb{R}^+}(E)$$

are equivalences of categories inverse to each other.

A cone in E is a subset $\gamma \subseteq E$ which is closed under the action of $\mathbb{R}^+$. We will call a cone convex (resp. proper) if all nonempty fibres $\gamma \cap \tau^{-1}(x)$ are convex (resp. proper).

Definition 2.4.5. For a subset $A \subseteq E$, its *polar set* $A^\circ \subseteq E^*$ is defined by

$$A^\circ = \{y \in E^* \mid \pi(y) \in \tau(A) \text{ and } \langle x, y\rangle \geq 0 \text{ for all } x \in A \cap \tau^{-1}(\pi(y))\}$$

The *antipodal set* A^a is the image of A under the antipodal mapping

$$a : E \to E, x \mapsto -x$$

Remark 2.4.6. For $A' \subseteq A$ with $\tau(A) = \tau(A')$, it is obvious that $A^\circ \subseteq A'^\circ$. One can also easily check that A° is always convex and that $A^{\circ\circ}$ is the (fibre-wise) closure of the convex hull of A.

Proposition 2.4.7 ([KS94, Lemma 3.7.10 and Prop. 3.7.12])**.** *Let* $F \in D^+_{\mathbb{R}^+}(E)$*.*

1. $\widehat{\widehat{F}} \cong F^a \otimes or_{E/Z}[-n]$.
2. Let $V \subseteq E^*$ be a convex open subset. Then:

$$R\Gamma(V, \widehat{F}) \cong R\Gamma_{V^\circ}(\tau^{-1}(\pi(V)), F) \cong R\Gamma_{V^\circ}(E, F)$$

3. Let $\eta \subseteq E^*$ be a closed convex cone containing the zero section. Then:

$$R\Gamma_\eta(E^*, \widehat{F}) \cong R\Gamma(\operatorname{Int}\eta^\circ, F) \otimes or_{E/Z}[-n],$$

   where $\operatorname{Int}\eta^\circ$ denotes the interior of η°.

4. For $F = A_\gamma$, where $\gamma \in E$ is a proper closed convex cone, we have

$$\widehat{A}_\gamma = A_{\operatorname{Int}\gamma^\circ}.$$

5. For $F = A_U$, where $U \subseteq E$ is a convex open cone $U \subseteq E$, we get

$$\widehat{A}_U = A_{U^{\circ a}} \otimes or_{E^*/Z}[-n].$$

6. There are isomorphisms

$$R\tau_! F \cong R\Gamma_X(F)\big|_X \cong R\pi_*(\widehat{F}) \cong (F)\big|_X$$
$$R\tau_* F \cong F\big|_X \cong R\pi_!(\widehat{F}) \otimes or_{E^*/X}\big|_X[n] \cong (R\Gamma_X(F) \otimes or_{E*/X})\big|_X$$

The Fourier-Sato transform behaves naturally with respect to pullbacks and vector bundle morphisms. It commutes with base change:

Proposition 2.4.8 ([KS94, Prop. 3.7.13]). *Let $f : Z' \to Z$ be a continuous map and $E' := E \times_Z Z'$ (resp. $E'^* := E^* \times_Z Z'$) the pullback vector bundles. Denote by f_τ (resp f_π) the natural projection $E' \to Z'$ of $E \to Z$ (resp $E'^* \to Z'$). Then there are natural isomorphisms for $F \in D^+_{\mathbb{R}^+}(A_E)$ and $G \in D^+_{\mathbb{R}^+}(A_{E'})$:*

$$\widehat{f_\tau^! F} \cong f_\pi^!(\widehat{F})$$
$$\widehat{f_\tau^{-1} F} \cong f_\pi^{-1}(\widehat{F})$$
$$\widecheck{Rf_{\tau *} G} \cong Rf_{\pi *}(\check{G})$$
$$\widecheck{Rf_{\tau !} G} \cong Rf_{\pi !}(\check{G})$$

Under a vector bundle morphism, it behaves as follows:

Proposition 2.4.9 ([KS94, Prop. 3.7.14-15]). *Let $f : E_1 \to E_2$ be a morphism of vector bundles over Z and let $f_d : E_2^* \to E_1^*$ be the dual morphism. For $F_i \in D^+_{\mathbb{R}^+}(A_{E_i}), i = 1, 2$ there are natural isomorphisms*

$$\begin{aligned} f_d^{-1}(\widehat{F}_1) &\cong \widehat{Rf_! F_1} \\ f_d^!(\widehat{F}_1) &\cong \widehat{Rf_* F_1} \otimes \omega_{E_2^*/E_1^*} \\ (\omega_{E_1/E_2} \widehat{\otimes f^{-1}} F_2) &\cong Rf_{d!}(\widehat{F}_2) \\ \widehat{f^! F_2} &\cong Rf_{d*}(\widehat{F}_2) \\ \widehat{F}_1 \boxtimes^L \widehat{F}_2 &\cong (F_1 \widehat{\boxtimes^L} F_2) \end{aligned}$$

2.5 Microlocalization

Let X be an n-manifold and $M \subseteq X$ a submanifold of codimension l. The normal bundle of $T_M X$ of M in X is the quotient bundle of rank l defined by the exact sequence

$$0 \to TM \to M \times_X TX \to T_M X \to 0.$$

We want to construct a new manifold $\tilde{X}_M$, the *normal deformation* of M in X, together with maps $p : \tilde{X}_M \to X$ and $t : \tilde{X}_M \to \mathbb{R}$ with the following properties:

1. $p^{-1}(X \backslash M)$ is isomorphic to $(X \backslash M) \times (\mathbb{R} \backslash \{0\})$.
2. $t^{-1}(\mathbb{R} \backslash \{0\})$ is isomorphic to $X \times (\mathbb{R} \backslash \{0\})$.
3. $t^{-1}(0)$ is isomorphic to $T_M X$.

Suppose $\phi_i : U_i \subseteq X \to \mathbb{R}^n$ is a local coordinate system with

$$M \cap U_i = \phi^{-1}(\{0\}^l \times \mathbb{R}^{n-l}).$$

For $x = (x', x'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}$ let

$$V_i = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} | (tx', x'') \in \phi_i(U_i)\}.$$

There are obvious projection maps $t_i : V_i \to \mathbb{R}$ and $p_i : V_i \to U_i$.

If $\phi_j : U_i \subseteq X \to \mathbb{R}^n$ is another such coordinate system, we define the gluing map

$$\psi_{ji} : V_i \times_{U_i} (U_i \cap U_j) \to \mathbb{R}^n$$

as follows: $\psi_{ji}(x, t) = (\psi'_{ji}(x, t), \psi''_{ji}(x, t))$ is the tuple satisfying

$$(t\psi'_{ji}(x, t), \psi''_{ji}(x, t)) = \phi_j(\phi_i^{-1}(tx', x'')).$$

This makes sense for $t = 0$, since $\phi_j \circ \phi_i^{-1}$ maps $\{0\}^l \times \mathbb{R}^{n-l}$ into itself. Taking a coordinate cover (U_i, ϕ_i) adapted to M as above, we define the normal deformation by

$$\tilde{X}_M := \left(\coprod_i V_i\right) / \sim,$$

where we identify $(t_i, x_i) \in V_i$ with $(t_j, x_j) \in V_j$ if $t_i = t_j$ and $x_j = \psi_{ji}(x_i, t_i)$.

$\tilde{X}_M$ is a manifold which is independent of the chosen atlas up to isomorphism. The maps t_i and p_i glue in an evident way to give maps t and p satisfying the properties one and two above. Let us show the third. On local patches we have

$$V_i \cap t^{-1}(0) = \mathbb{R}^l \times (\phi_i(U_j \cap M)),$$

identifying $\{0\}^l \times \mathbb{R}^{n-l}$ with $\mathbb{R}^{n-l}$. Letting $\psi_{ji}(x,t) = (\psi'_{ji}, \psi''_{ji})$ and $\phi_j \circ \phi_i^{-1} = (\phi'_{ji}, \phi''_{ji})$, the gluing maps look like

$$\psi'_{ji}(X, 0) = \sum_{k=1}^{l} x_k \frac{\partial}{\partial x_k} \phi'_{ji}(0, x'')$$
$$\psi''_{ji}(X, 0) = \phi''_{ji}(0, x''),$$

which are just the transition maps for the normal bundle $T_M X$.

Remark 2.5.1. There is a natural $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ action on $\tilde{X}_M$, given in local coordinates by

$$c \cdot (x', x'', t) = (cx', x'', c^{-1}t).$$

This action restricts to the scaling action on $T_M X = t^{-1}(0)$.

Let $\Omega = t^{-1}(\mathbb{R}^+)$. The various maps fit into the diagram:

$$\begin{tikzcd} T_M X \arrow[r, hook, "s"] \arrow[d, "\tau"] & \tilde{X}_M \arrow[d, "p"] & \Omega \arrow[l, hook', "j"'] \arrow[dl, "\tilde{p}"] \\ M \arrow[r, hook, "i"] & X & \end{tikzcd}$$

Definition 2.5.2. Let $S \subseteq X$. The *normal cone* to S along M is the closed, conic subset of $T_M X$ defined by

$$C_M(S) := T_M X \cap \overline{\tilde{p}^{-1}(S)}.$$

Definition 2.5.3. Let $F \in D^b(A_X)$. The *specialization* $\nu_M(F)$ of F is defined as

$$\nu_M(F) = s^{-1} R j_* \tilde{p}^{-1} F$$

The (derived) sections of $\nu_M(F)$ over an open set $V \subseteq T_M X$ are given by sections of F over $U \subseteq X$ such that $C_M(U)$ asymptotically approaches V. More precisely, we have the following:

Theorem 2.5.4 ([KS94, Thm 4.2.3])**.** *Let $F \in D^b(A_X)$.*

1. $\nu_M(F) \in D^b_{\mathbb{R}^+}(T_M X)$ *and* $\mathrm{supp}(\nu_M(F)) \subseteq C_M(\mathrm{supp}\, F)$.

2. *Let* $V \subseteq T_M X$ *be open and conic. Then*

$$H^j(V, \nu_M(F)) = \varinjlim_{U} H^j(U, F),$$

where the colimit runs over open sets $U \subseteq X$*, such that* $C_M(X \backslash U) \cap V = \emptyset$.

3. *Let* $A \subseteq T_M X$ *be closed and conic. Then*

$$H^j_A(T_M X, \nu_M(F)) = \varinjlim_{Z,U} H^j_{Z\cap U}(U, F),$$

where the colimit ranges through open neighbourhoods $U \subseteq X$ *of* M *and closed sets* $Z \subseteq X$ *such that* $C_M(Z) \subseteq A$.

4. *There are isomorphisms*

$$\nu_M(F)\big|_M \cong R\tau_*(\nu_M(F)) \cong F\big|_M$$
$$R\Gamma_M(\nu_M(F))\big|_M \cong R\tau_!(\nu_M(F)) \cong R\Gamma_M(F)\big|_M$$

5. *Let* $\mathring{\tau} : T_M X \backslash M \to M$*. Then*

$$R\mathring{\tau}_M(\nu_M(F)) \cong R\Gamma_{X\backslash M}(F)\big|_M.$$

Now let $f : Y \to X$ be a smooth map of manifolds and $N \subseteq Y$ a closed submanifold with $f(N) \subseteq M$. The differential of f descends to a map

$$T_N f : T_N Y \to T_M X.$$

Proposition 2.5.5 ([KS94, Prop. 4.2.4-6])**.** *Let* $G \in D^b(A_Y)$ *and* $F \in D^b(A_X)$.

1. *There is a diagram of canonical morphism*

$$\begin{array}{ccc} R(T_N f)_! \nu_N(G) & \longrightarrow & \nu_M(G)(Rf_! G) \\ \downarrow & & \downarrow \\ R(T_N f)_* \nu_N(G) & \longleftarrow & \nu_M(G)(Rf_* G) \end{array}$$

These maps are isomorphism if the maps $f|_{\mathrm{supp}(G)}$ *and* $T_N f|_{C_N(\mathrm{supp}(G))}$ *are proper and* $\mathrm{supp}(G) \cap f^{-1}(M) \subseteq N$.

2. *There are canonical morphisms*

$$\alpha : (T_N f)^{-1}\nu_M(F) \to \nu_N(f^{-1}F)$$
$$\beta : \nu_N(f^! F) \to (T_N f)^! \nu_M(F)$$

These maps are isomorphisms if $f : Y \to X$ *and* $f\big|_N : N \to M$ *are smooth.*

3. *There is a natural morphism in* $D^b(T_{M\times N}X \times Y)$

$$\nu_M(F) \boxtimes^L \nu_N(G) \to \nu_{M\times N}(F \boxtimes^L G).$$

Remark 2.5.6. We have a canonical isomorphism

$$\omega_{T_N Y/N\times T_M X} \cong \omega_{Y/X} \otimes \omega_{N/M}^{-1}.$$

Then the morphism α and β fit into the commutative diagram

$$\begin{array}{ccc} \omega_{T_N Y/T_M X} \otimes (T_N f)^{-1}\nu_M(F) & \xrightarrow[\omega_{N/M}\otimes\alpha]{} & \nu_N(\omega_{Y/X} \otimes f^{-1}F) \\ \downarrow & & \downarrow \\ (T_N f)^! \nu_M(F) & \xleftarrow[\beta]{} & \nu_N(f^! F) \end{array}$$

where the vertical morphisms are deduced from Thm. 2.2.3. See [KS94, Section 4.2].

Now let $T^*_M X$ be the conormal bundle of M in X, i.e. the kernel of the restriction map $M \times_X T^*X \to T^*M$. We identify M and the zero section $i : M \to T^*_M X$ of $T^*_M X$. Denote by $\pi : T_M X \to M$ the projection and by $\mathring{\pi} : T_M X \backslash M \to M$ the restriction to the complement of the zero section.

Definition 2.5.7. Let $F \in D^b(A_X)$. The *microlocalization* of F along M is the Fourier-Sato transform of its specialization:

$$\mu_M(F) := \widehat{\nu_M(F)}.$$

Theorem 2.5.8. *Let* $F \in D^b(A_X)$.

1. $\mu_M(F) \in D^b_{\mathbb{R}^+}(A_{T^*_M X})$

2. *For a convex open* $V \subseteq T^*_M X$, *the sheaf cohomology groups are given by*

$$H^j(V, \mu_M(F)) = \varinjlim_{U,Z} H^j_{Z\cap U}(U, F),$$

where U *ranges through the open neighbourhoods of* $\pi(V) \subseteq M$ *and* Z *through the closed subsets such that* $C_M(Z) \subseteq V^\circ$.

3. *For* $Z \subseteq T^*_M X$ *a proper closed convex cone containing the zero section, we get:*

$$H^j_Z(T^*_M X, \mu_M(F) \otimes or_{M/X}) = \varinjlim_{U} H^{j-l}(U, F),$$

where U *ranges through the family of open subsets of* X *such that*

$$C_M(X\backslash U) \cap \operatorname{Int} Z^{\circ a} = \emptyset.$$

4. There are natural isomorphisms

$$\mu_M(F)\big|_M \cong R\pi_*\mu_M(F) \cong R\Gamma_M(F)\big|_M \cong i^! F$$
$$R\pi_!\mu_M(F) \cong R\Gamma_M(\mu_M(F)) \cong i^{-1}F \otimes \omega_{M/X}.$$

Proof. This follows directly from Theorem 2.5.4 and Proposition 2.4.7. □

Let again $f : Y \to X$ a morphism of manifolds and $N \subseteq Y$ a submanifold with $f(N) \subseteq M$. The pullback $f^* : T^*_M X \to T^*_N Y$ factors as

$$T^*_N Y \xleftarrow{f_d} N \times_M T^*X \xrightarrow{f_\pi} T^*_M X,$$

i.e. we have $f_d = f^* \circ f_\pi$. The microlocalization behaves functorially as follows:

Proposition 2.5.9. *Let $G \in D^b(A_Y)$ and $F \in D^b(A_X)$.*

1. There exists a canonical commutative diagram

$$\begin{tikzcd} Rf_{\pi!}f_d^{-1}\mu_N(G) \arrow[r] \arrow[d] & \mu_M(Rf_!G) \arrow[d] \\ Rf_{\pi*}(f_d^!\mu_N(G) \otimes \omega_{Y/X} \otimes \omega_{N/M}^{-1}) & \mu_M(Rf_*G) \arrow[l] \end{tikzcd}$$

If $\operatorname{supp}(G) \to X$ and $C_N(\operatorname{supp}(G)) \to T_M X$ are proper and $f^{-1}(M) \cap \operatorname{supp}(G) \subseteq N$, then these maps are isomorphisms.

2. There exists a canonical commutative diagram

$$\begin{tikzcd} Rf_{d!}(\omega_{N/M} \otimes f_\pi^{-1}\mu_M(F)) \arrow[r] \arrow[d] & \mu_N(\omega_{Y/X} \otimes f^{-1}F) \arrow[d] \\ Rf_{d*}f_\pi^!\mu_M(F) & \mu_N(f^!F) \arrow[l] \end{tikzcd}$$

These maps are isomorphisms if $f : Y \to Y$ and $f\big|_N : N \to M$ are submersions.

3. There is a natural morphism in $D^b_{\mathbb{R}^+}(A_{T^*_{M\times N}X\times Y})$:

$$\mu_M(F) \boxtimes^L \mu_N(G) \to \mu_{M\times N}(F \boxtimes^L G).$$

Proof. The maps are deduced from Prop. 2.5.5 by using the functoriality of the Fourier-Sato transform (Prop. 2.4.8 and 2.4.9). We refer to [KS94, Prop.4.3.4-6] for further details.

□

The functor $\mu_N(\cdot)$ can be viewed as a microlocal refinement of the functor $R\Gamma_M(\cdot) \cong R\mathcal{H}om(A_M, \cdot)$. We can generalize this to a microlocal version of $R\mathcal{H}om(\cdot, \cdot)$.

Let $\Delta_X \subseteq X \times X$ be the diagonal. The projection $T^*(X \times X) \to T^*X$ onto the second factor induces an isomorphism $T^*_{\Delta_X}(X \times X) \cong T^*X$. Let $q_1, q_2 : X \times X \to X$ be the projections to the first (resp. second) factor.

Definition 2.5.10. Let $F, G \in D^b(A_X)$. The relative microlocalization functor is given by:

$$\mu hom(G, F) = \mu_{\Delta_X}(Rhom(q_2^{-1}G, q_1^! F)).$$

Remark 2.5.11. Our definition is actually a special case of the μhom-functors defined in [KS94], which will be enough for our purposes.

Note that $\mu hom(G, F)$ is a complex of sheaves on $T^*_{\Delta_X}(X \times X) \cong T^*X$. We denote by π the projection

$$\pi : T^*_{\Delta_X}(X \times X) \to \Delta_X \cong X$$

The next proposition shows that the μhom functor is indeed a microlocal refinement of the usual morphism functor.

Proposition 2.5.12. *There is a canonical isomorphism*

$$R\pi_* \mu hom(G, F) \cong R\mathcal{H}om(G, F).$$

Proof. Thm. 2.2.3 and Thm. 2.5.8 give the isomorphisms

$$\begin{aligned} R\pi_* \mu hom(G, F) &\cong Rq_{2*} R\Gamma_{\Delta_X} R\mathcal{H}om(q_2^{-1}G, q_1^! F) \\ &\cong \delta^! R\mathcal{H}om(q_2^{-1}G, q_1^! F) \\ &\cong R\mathcal{H}om(\delta^{-1} q_2^{-1} G, \delta^! q_1^! F) \\ &\cong R\mathcal{H}om(G, F), \end{aligned}$$

where $\delta : X \cong \Delta_X \to X \times X$ is the inclusion. $\square$

We recover the usual microlocalization by setting $G = A_M$.

Proposition 2.5.13 ([KS94, Prop. 4.4.3])**.** *Let $f : M \to X$ be the inclusion of a closed submanifold and $j : T^*_M X \hookrightarrow T^*X$ the induced embedding. Then one has a canonical identification*

$$\mu hom(A_M, F) \cong j_* \mu_M(F).$$

The following functoriality property will play a major part in out treatment of boundary value distributions.

Proposition 2.5.14 ([KS94, Prop. 4.4.7])**.** *Let $f : Y \to X$ a morphism of manifolds and $F_1, F_2 \in D^b(A_X)$. There is a natural morphism*

$$Rf_{d!} f_\pi^{-1} \mu hom(F_1, F_2) \to \mu hom(f^! F_1, f^{-1} F_2 \otimes \omega_{Y/X}).$$

2.6 Micro-support of sheaves

The microlocalization functor gives us a microlocal refinement of the support of sections of a fixed sheaf. One category level higher, Kashiwara and Schapira also gave a microlocal refinement of the support of sheaves.

Let $F \in D^b(A_X)$ and consider the following condition for a point $(x_0, \xi_0) \in T^*X$: There is an open neighbourhood $U \subseteq T^*X$ of (x_0, ξ_0) such that for every $x_1 \in \pi(U)$ and every C^1-function $\phi : V \to \mathbb{R}$ defined in a neighbourhood $V \subseteq \pi(U)$ of x_1 such that

1. $\phi(x_1) = 0$
2. $d\phi(x_1) \in U$,

we have

$$(R\Gamma_{\{\phi(x) \geq 0\}}(F))_{x_1} = 0.$$

Definition 2.6.1. The *micro-support* $SS(F) \subseteq T^*X$ of F is the complement of all points $(x_0, \xi_0) \in T^*X$ satisfying the above condition.

To develop an intuition for the above definition, consider the case that $X \subseteq V$ is an open set in a real vector space V and $(x, \xi) \in T^*X \cong X \times V^*$. If $(x_0, \xi_0) \notin SS(F)$, then we can choose $\phi(x) = \langle \xi_0, x - x_0 \rangle$. We have the distinguished triangle

$$0 \cong (R\Gamma_{\{\langle \xi_0, x - x_0 \rangle \geq 0\}} F)_{x_0} \longrightarrow F_{x_0} \longrightarrow (R\Gamma_{\{\langle \xi_0, x - x_0 \rangle < 0\}} F)_{x_0} \xrightarrow{+1}$$

i.e. the restriction map $F \to R\Gamma_{\{\langle \xi_0, x - x_0 \rangle < 0\}} F$ is an isomorphism in a neighbourhood of x_0. Hence we can think of points $(x_0, \xi_0) \in SS(F)$ as directions where interesting things are happening.

The following basic properties of the micro-support follow easily from the definition.

Proposition 2.6.2 ([KS94, Prop. 5.1.3])**.** *The micro-support $SS(F)$ is a closed conic subset of T^*X with the following properties:*

1. $SS(F) \cap T^*X = \mathrm{supp}(F)$
2. $SS(F) = SS(F[1])$
3. *For a distinguished triangle*

$$F_1 \longrightarrow F_2 \longrightarrow F_3 \xrightarrow{+1} \quad ,$$

we have the inclusions

$$SS(F_i) \subseteq SS(F_j) \cup SS(F_k)$$
$$(SS(F_i) \backslash SS(F_j)) \cup (SS(F_i) \backslash SS(F_j)) \subseteq SS(F_k),$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Most calculations of the micro-support are based on the following:

Proposition 2.6.3 ([KS94, Prop. 5.3.1])**.** *Let $\gamma \subseteq V$ be a closed convex cone in a finite-dimensional real vector space V. The micro-support of A_γ over the point $0 \in \gamma$ is*

$$SS(A_\gamma) \cap \pi^{-1}(0) = \gamma^\circ.$$

Example 2.6.4. Let $M \subseteq X$ a closed submanifold and $F = A_M$. Since $F_x = 0$ for $x \notin M$ it suffices to consider the case $x \in M$. The micro-support can be calculated locally, so we can choose coordinates and assume that $M = W \hookrightarrow V$ is the inclusion of a linear subspace and $x = 0$. Then the above proposition applies with $\gamma = W$ and we get

$$SS(A_W) \cap \pi^{-1}(0) = W^\perp.$$

under the identification $T^*V = V \times V^*$. Varying $x \in M$, we obtain

$$SS(A_M) = T^*_M X.$$

Example 2.6.5. Let $\phi : X \to \mathbb{R}$ be a C^1-function, such that $0 \in \mathbb{R}$ is a regular value of ϕ and $F = A_{\{\phi(x) \geq 0\}}$. As above, we can assume that X is a vector space, $\phi(x) = x_1$ and $x_0 = 0$. Set $\gamma = \{\phi(x) \geq 0\}$. The dual cone can then be identified with the ray $\mathbb{R}_{\geq 0}\mathrm{d}\phi(x_0)$ and we obtain

$$SS(A_{\{\phi(x)\geq 0\}}) = \{(x, \lambda \mathrm{d}\phi(x)) \mid \lambda\phi(x) = 0, \lambda \geq 0, \phi(x) \geq 0\}.$$

From the exact sequence

$$0 \longrightarrow A_{\{\phi(x)<0\}} \longrightarrow A_X \longrightarrow A_{\{\phi(x)\geq 0\}} \longrightarrow 0$$

and $SS(A_X) = T^*_X X$ we obtain

$$SS(A_{\{\phi(x)<0\}}) = \{(x, \lambda \mathrm{d}\phi(x)) \mid \lambda\phi(x) = 0, \lambda \geq 0, \phi(x) \leq 0\}.$$

It will rarely be possible to compute the micro-support exactly, but upper bounds will often be enough.

Proposition 2.6.6 ([KS94, Prop. 5.4.1 and 5.4.2])**.** *Let X_1, X_2 be two manifolds and $q_i : X_1 \times X_2 \to X_i, i = 1, 2$ the natural projections. For $F_i \in D^b(A_{X_i})$, there are inclusions*

$$SS(F_1 \boxtimes F_2) \subseteq SS(F_1) \times SS(F_2)$$
$$SS(R\mathcal{H}om(q_1^{-1}F_1, q_2^{-1}F_2)) \subseteq SS(F_1) \times SS(F_2)^a$$

Let $f : Y \to X$ a morphism of manifolds. Recall that we have the decomposition

$$T^*Y \xleftarrow{f_d} Y \times_X T^*X \xrightarrow{f_\pi} T^*X$$

of the pullback map $f^* : T^*X \to T^*Y$.

Proposition 2.6.7 ([KS94, Prop. 5.4.4])**.** *Suppose $G \in D^b(A_Y)$ such that $f : Y \to X$ is proper on supp(G). Then*

$$SS(Rf_*G) \subseteq f_\pi(f_d^{-1}(SS(G))).$$

For inverse images the following relative smoothness hypothesis will play an important role:

Definition 2.6.8. Let $A \subseteq T^*X$ be a closed conic subset. The map f is called *non-characteristic* for A if $f_d : f_\pi^{-1}(A) \to T^*Y$ is proper. If A is the micro-support of a complex of sheaves $F \in D^b(A_X)$, then we say that f is non-characteristic for F.

A more intuitive way to phrase this condition is the following:

Proposition 2.6.9. *f is non-characteristic for a closed conic subset $A \subseteq T^*X$ if and only if*

$$\ker f_d \cap f_\pi^{-1}(A) \subseteq Y \times_X T_X^*X.$$

Proof. The condition is local on Y so we can reduce it to the following statement about bundle maps: Let K be a compact Hausdorff space,

$$L : V \times K \to W \times K$$

a map of finite-dimensional real vector vector bundles over K and $A \subseteq V \times K$ a closed set with conic fibres. Then $L\big|_A$ is proper if and only if

$$\ker L \cap A \subseteq \{0\} \times K := 0_K.$$

The only if direction is clear: The set $\ker L \cap A = L^{-1}(0_K) \cap A$ has conic fibres and therefore can only be compact if it is contained in 0_K. Now suppose $\ker L \cap A$ is contained in the image of the zero section. Choose continuously varying norms $|\cdot|_V$ (resp. $|\cdot|_W$) on $V \times K$ (resp. $W \times K$). Let S_V be the unit sphere bundle corresponding to $|\cdot|_V$. The hypothesis imply that

$$\delta := \min_{v \in A \cap S_V} |L(v)|_W > 0.$$

Since A is conic, we get

$$|Lv|_W \geq \delta |v|_V$$

for all $v \in A$. Hence the preimage $L^{-1}(B) \cap A$ of a bounded set $B \subseteq W \times K$ stays bounded and L_A is proper. □

Example 2.6.10. If $f : Y \to X$ is smooth, then f is non-characteristic for all conic subsets $A \subseteq T^*X$.

Example 2.6.11. If $A = SS(A_M) = T_M^*X$ for a closed submanifold $M \subseteq X$, then f is non-characteristic for A if and only if f is transverse to M. Recall that the later means that $\mathrm{d}f(T_yY) + T_{f(y)}M = T_{f(y)}X$ for $y \in f^{-1}(M)$. Taking duals, this is equivalent to the injectivity of

$$T_{f(y)}^*X \to T_y^*Y \oplus T_{f(y)}^*M,$$

which means that f is non-characteristic for $SS(A_M)$ by Example 2.6.4 and Prop. 2.6.9.

Example 2.6.12. Suppose $F, G \in D^b(A_X)$ and let $\Delta : X \to X \times X$ be the diagonal embedding. Then Δ is non-characteristic for $F \boxtimes G$ if $SS(F) \cap SS(G)^a \subseteq T^*_X X$.

Recall that the natural map $f^{-1}F \otimes \omega_{Y/X} \to f^! F$ is an isomorphism if f is smooth. We can now state the following important generalization.

Proposition 2.6.13 ([KS94, Prop. 5.4.13])**.** *Suppose $f : Y \to X$ is non-characteristic for $F \in D^b(A_X)$. Then*

$$SS(f^{-1}F) \subseteq f_d(f_\pi^{-1}(SS(F)))$$

and the natural map

$$f^{-1}F \otimes \omega_{Y/X} \to f^! F$$

is an isomorphism.

Applying the above result with $f = \Delta : X \to X \times X$ and using Example 2.6.12 gives the following.

Proposition 2.6.14 ([KS94, Prop. 5.4.14])**.** *Let $F, G \in D^b(A_X)$.*

1. *Suppose $SS(F) \cap SS(G)^a \subseteq T^*_X X$. Then*

$$SS(F \otimes^L G) \subseteq SS(F) + SS(G).$$

2. *Suppose $SS(F) \cap SS(G) \subseteq T^*_X X$. Then*

$$SS(R\mathcal{H}om(F, G)) \subseteq SS(G) + SS(F)^a.$$

Example 2.6.15. For $D'(F) := R\mathcal{H}om(F, A_X)$, we get

$$SS(D'(F)) \subseteq SS(F)^a.$$

The micro-support is naturally related to the microlocalization μhom as follows:

Proposition 2.6.16 ([KS94, Cor. 5.4.10])**.** *Let $F, G \in D^b(A_X)$. Then*

$$supp(\mu hom(F, G)) \subseteq SS(G) \cap SS(F).$$

In particular, if $F = \mathbb{C}_M$ for $M \subseteq X$ a closed submanifold, then

$$supp(\mu_M(G)) \subseteq T^*_M X \cap SS(G).$$

2.7 Subanalytic sheaves and stratifications

For a real analytic manifold X, we can define an especially nice class of locally closed subsets, which are stable under most natural operations, while their topology is relatively tame.

Definition 2.7.1. A subset $Z \subseteq X$ is *subanalytic* at $x \in Z$ if there is an open neighbourhood U of x and morphisms $f_j : Y_j \to X, g_j : \tilde{Y}_j \to X, j = 1, \ldots, N$ of real analytic manifolds, such that

$$U \cap Z = U \cap \bigcup_{j=1}^{N} (f_j(Y_j) \backslash g(\tilde{Y}_j)).$$

Z is called *subanalytic* if it is subanalytic at every point of X.

The main properties of subanalytic subsets are summarized in the following.

Proposition 2.7.2 ([BM88]). *Let $Z, Z_1, Z_2 \subseteq X$ be subanalytic subsets of X, $f : Y \to X$ a morphisms of real analytic manifolds and $W \subseteq Y$ a subanalytic subset of Y.*

1. *The closure $\overline{Z}$ and interior* $\mathrm{Int}(Z)$ *are subanalytic. The connected components of Z are subanalytic and locally finite.*
2. *The subsets $Z_1 \cup Z_2, Z_1 \cap Z_2$ and $Z_1 \backslash Z_2$ are subanalytic.*
3. *The inverse image $f^{-1}(Z) \subseteq Y$ is subanalytic. If $f|_{\overline{W}}$ is proper, then $f(W)$ is subanalytic.*
4. *If Z is closed in X, then there exists a proper map $g : \tilde{Y} \to X$ of real analytic manifolds with $g(\tilde{Y}) = Z$.*
5. *Every subanalytic subset $Z \subseteq X$ has a dense open subset $Z_{reg} \subseteq Z$, such that $Z_{reg} \subseteq X$ is an analytic submanifold.*

If $Z \subseteq X$ is subanalytic, then $Z_{sing} = Z \backslash Z_{reg}$ has strictly lower dimension. By induction, we can express Z as a disjoint union of subanalytic submanifolds.

Definition 2.7.3. Let $Z \subseteq X$ be a subanalytic subset. A locally finite decomposition $U = \bigsqcup_{\alpha \in I} X_\alpha$ is called a *subanalytic stratification* if the X_α are subanalytic submanifolds of X and for every $(\alpha, \beta) \in I^2$, $X_\alpha \cap \overline{X}_\beta \neq \emptyset$ implies $X_\alpha \subseteq \overline{X}_\beta$.

Definition 2.7.4. A sheaf $F \in \mathrm{Mod}(A_X)$ is called $\mathbb{R}$-*constructible*, if there is a subanalytic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$, such that the restrictions $F\big|_{X_\alpha}$ are locally constant sheaves of finite rank.

Example 2.7.5. If $Z \subseteq X$ is subanalytic, then $\mathbb{A}_Z$ is constructible. In fact, one can always construct a subanalytic triangulation of Z ([KS94, Prop. 8.2.5]).

We denote by $\mathbb{R} - Cons(X) \subseteq \mathrm{Mod}(A_X)$ the subcategory of $\mathbb{R}$-constructible sheaves and by $D^b_{\mathbb{R}-c}(A_X) \subseteq D^b(A_X)$ the subcategory of sheaf complexes with $\mathbb{R}$-constructible cohomologies.

Theorem 2.7.6 ([KS94, Thm. 8.4.5])**.** *The natural map $D^b(\mathbb{R}-Cons(X)) \to D^b_{\mathbb{R}-c}(A_X)$ is an equivalence of categories.*

The category $D^b_{\mathbb{R}-c}(A_X)$ is essentially stable under Grothendiecks six functor formalism:

Theorem 2.7.7 ([KS94, Prop. 8.4.6])**.** *Let $f : Y \to X$ be an analytic map and $F, F_1, F_2 \in D^b_{\mathbb{R}-c}(A_Y), G \in D^b_{\mathbb{R}-c}(A_X)$.*

1. *The complexes $R\mathcal{H}om(F_1, F_2)$ and $F_1 \otimes^L F_2$ belong to $D^b_{\mathbb{R}-c}(A_Y)$.*
2. *$f^{-1}G$ and $f^!G$ belong to $D^b_{\mathbb{R}-c}(A_X)$.*
3. *If f is proper on $\mathrm{supp}(F)$, then $Rf_*F \cong Rf_!F$ belongs to $D^b_{\mathbb{R}-c}(A_X)$.*

The subcategory $D^b_{\mathbb{R}-c}(A_X) \subseteq D^b(A_X)$ can be characterized by the micro-support.

Theorem 2.7.8 ([KS94, Prop. 8.4.1])**.** *Let $F \in D^b(A_X)$. The following are equivalent:*

1. *$F \in D^b_{\mathbb{R}-c}(A_X)$.*
2. *There exists a subanalytic stratification $X = \bigsqcup_{\alpha \in I} X_\alpha$, such that*

$$SS(F) \subseteq \bigsqcup_{\alpha \in I} T^*_{X_\alpha} X.$$

and the cohomology sheaves $H^j(F)_x$ for $j \in \mathbb{Z}$ and $x \in X$ are finitely generated.

The following microlocal refinement of the classical Sard theorem will be useful in the sequel.

Proposition 2.7.9 ([KS94, 8.3.12])**.** *Let $F \in D^b_{\mathbb{R}-c}(A_X)$ and $\varphi : X \to \mathbb{R}$ be an analytic map, which is proper on $\mathrm{supp}(F)$. Then the image $\varphi(S) \subseteq \mathbb{R}$ of the set*

$$S = \{x \in X \mid \mathrm{d}\varphi(x) \in SS(F)\}$$

is discrete.

3 Toric varieties

We will give a very brief review of the theory of toric varieties. Our presentation is based on [CLS11] and in part adapted from the article [Sch18]. At the end, we also give a toric approach to wonderful model compactifications based on [FK04] and review the theory of generalized permutahedra ([AA17], [Pos09], [PRW06]), which will give provide us with a convenient framework to investigate the momentum and parametric representations of Feynman amplitudes.

3.1 Polyhedra and polytopes

Let us first review some basic concepts of polyhedral geometry. We refer the reader to [Zie95] for details and proofs.

Let V be a finite-dimensional real vector space and V^* its dual. A *polyhedron* is a subset of V, which is given as an intersection of finitely many affine half-spaces, i.e. it is a subset of the form

$$P = \{p \in V \mid \langle p, u_i \rangle \geq d_i, i = 1, \dots, s\},$$

where $u_i \in V^*$ and $d_i \in \mathbb{R}$. The *affine span* H_P of a polyhedron P is the smallest affine subspace of V containing P. Any weight vector $u \in V^*$ defines the face

$$F_u P = \{p \in P \mid \langle p, u \rangle = \min_{\tilde{p} \in P} \langle \tilde{p}, u \rangle\}$$

We denote the set of dimension k faces of P by $P(k)$. A face $F = F_u P$ of codimension one is called a *facet* and a face of dimension 0 is called a *vertex*. Any polyhedron of dimension n then has an irredundant presentation

$$P = \{p \in H_P \mid \langle p, u_F \rangle \geq d_F, F \in P(n-1)\},$$

where u_F is the weight vector defining the facet F.

A *polyhedral cone* is a polyhedron of the form

$$\sigma = \{p \in V \mid \langle p, u_i \rangle \geq 0, i = 1, \dots, s\}.$$

Alternatively, we can describe σ as the positive hull of finitely many vectors $v_1, \dots, v_r \in V$:

$$\sigma = \text{pos}(v_1, \dots, v_r) := \{p \in V \mid p = \sum_{i=1}^{r} \lambda_i v_i, \lambda_i \geq 0\}.$$

A polyhedral cone is called *strongly convex* or *proper*, if it does not contain a subspace of positive dimension, i.e. if $0 \in \sigma$ is a vertex. Every polyhedral cone $\sigma \subseteq V$ defines a dual cone

$$\sigma^\vee = \{u \in V^* \mid \langle v, u\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

A bounded polyhedron $P \subseteq V$ is called a *polytope*. Every polytope can be described as the convex hull of its vertices $v \in P(0)$, i.e.

$$\begin{aligned} P &= \mathrm{Conv}(v \mid v \in P(0)) \\ &:= \{p \in V \mid p = \sum_{v\in P(0)} \lambda_v v,\ \lambda_v \geq 0,\ \sum_{v \in P(0)} \lambda_v = 1\}. \end{aligned}$$

For two polytopes $P, Q \subseteq V$, their Minkowski sum is defined as

$$P + Q = \{p \in V \mid p = p_1 + p_2 \text{ for } p_1 \in P, p_2 \in Q\}.$$

For $r \in (0, \infty)$, we also define the scaled polytope

$$rP = \{rp \in V \mid p \in P\}.$$

3.2 Cones and fans.

Let N be a lattice, i.e. a free abelian group of finite rank $n = \mathrm{rk}\, N$. Its dual lattice is $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The algebraic torus associated to N is $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong G_m^n(\mathbb{C})$. Elements $m \in M$ define characters $t^m \in \mathrm{Hom}_{\mathbb{Z}}(T_N, \mathbb{C}^*)$. Under this identification, the coordinate ring of T_N is the ring of Laurent polynomials

$$\mathcal{O}(T_N) = \mathbb{C}[M].$$

Similarly, elements $u \in N$ define one-parameter subgroups

$$\mathbb{C}^* \to T_N, \quad \lambda \mapsto u \otimes \lambda.$$

Definition 3.2.1. A complex variety X is *toric* if it has an action of a torus T_N with a dense torus orbit.

We will only consider *normal* toric varieties. They can be completely described by certain polyhedral data. Let $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex polyhedral cone. σ is *rational* if there are lattice elements $u_1, \ldots, u_s \in N$, such that

$$\sigma = \mathrm{pos}(u_1, \ldots, u_s).$$

Its dual cone

$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}$$

is again a rational polyhedral cone and the set $S_\sigma = M \cap \sigma^\vee$ is a finitely generated semigroup. The toric variety associated to σ is $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$. The torus action is given in terms of the coordinate rings by

$$\Delta : \mathbb{C}[S_\sigma] \to \mathbb{C}[M] \otimes \mathbb{C}[S_\sigma], \quad t^m \mapsto t^m \otimes t^m.$$

The inclusion $T_N \hookrightarrow X_\sigma$ of the dense torus orbit is dually given by the map $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[M]$.

We can construct all normal toric varieties by gluing such affine varieties along open subsets, as long as the corresponding cones intersect nicely.

Definition 3.2.2. A *fan* Σ in $N_\mathbb{R}$ is a collection of rational, strongly convex polyhedral cones $\sigma \subseteq N_\mathbb{R}$ such that:

1. If $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ is a face of σ then $\tau \in \Sigma$.
2. Two cones $\sigma_1, \sigma_2 \in \Sigma$ are either disjoint or intersect in a common face $\tau \in \Sigma$.

We call Σ a *generalized fan* if it consists of rational polyhedral cones which are not necessarily strongly convex.

The set of cones of dimension k is denoted by $\Sigma(k)$. A one-dimensional cone $\rho \in \Sigma(1)$ is called a *ray*. The union $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_\mathbb{R}$ is the *support* of Σ. Each (generalized) fan naturally has the structure of a partially ordered set, where $\tau \preceq \sigma$ if and only if τ is a face of σ.

If σ_1, σ_2 are two rational, strongly convex cones intersecting in the common face $\tau = \sigma_1 \cap \sigma_2$, then the dual inclusions $\sigma_1^\vee \subseteq \tau^\vee \supseteq \sigma_2^\vee$ define the inclusions

$$\mathbb{C}[S_{\sigma_1}] \hookrightarrow \mathbb{C}[S_\tau] \hookleftarrow \mathbb{C}[S_{\sigma_2}].$$

One can show that $\mathbb{C}[S_\tau]$ is a common localization of $\mathbb{C}[S_{\sigma_i}]$, such that $U_\tau \subseteq U_{\sigma_i}$. Gluing U_{σ_1} and U_{σ_2} along the dense open subset U_τ gives a new toric variety. This can be done coherently for all cones in a fan and we obtain the following:

Proposition 3.2.3 ([CLS11, Thm. 3.1.5])**.** *If Σ is a fan in $N_\mathbb{R}$, then the U_σ for $\sigma \in \Sigma$ glue together to give a normal toric variety X_Σ and every normal toric variety is of this form up to isomorphism.*

Example 3.2.4. If $\sigma \subseteq N_\mathbb{R}$ is a rational, strongly convex polyhedral cone, then the collection of faces $\tau \subseteq \sigma$ form fan which we also denote by σ. The corresponding toric variety is just U_σ.

Properties of X_Σ are reflected in properties of the fan:

Proposition 3.2.5 ([CLS11, Thm. 3.4.1 and 3.1.19])**.** *Let X_Σ be a toric variety with fan Σ.*

1. *X_Σ is complete if $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma = N_\mathbb{R}$.*

2. X_Σ *is smooth if and only if every cone* $\sigma \in \Sigma$ *can be generated by part of a* $\mathbb{Z}$*-basis of* N*. In this case the fan* Σ *and its cones* $\sigma \in \Sigma$ *are called smooth.*

Definition 3.2.6. Let $X_{\Sigma_1}, X_{\Sigma_2}$ be toric varieties with fans Σ_i in $(N_i)_\mathbb{R}$. A morphism $\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$ is *toric* if $\varphi|_{T_{N_1}}$ induces a group morphism $T_{N_1} \to T_{N_2}$.

Being toric automatically implies that φ is T_{N_i}-equivariant. We can identify N_i with the one-parameter subgroups of T_{N_i} and since φ is a group morphism, we get a homomorphism

$$\overline{\varphi} : N_1 \to N_2$$

of lattices. We say that such a morphism is compatible with the fans Σ_i if for every $\sigma_1 \in \Sigma_1$ there is $\sigma_2 \in \Sigma_2$ with $\overline{\varphi}(\sigma_1) \subseteq \sigma_2$.

Proposition 3.2.7 ([CLS11, Thm 3.3.4])**.** *If* $\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$ *is toric, then the induced map* $\varphi : N_1 \to N_2$ *is compatible with the fans* Σ_1, Σ_2*.*

Conversely, every morphism $\overline{\varphi} : N_1 \to N_2$ *compatible with the fans uniquely determines a toric morphism* $\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$ *which extends*

$$\overline{\varphi} \otimes 1 : N_1 \otimes \mathbb{C}^* = T_{N_1} \to N_2 \otimes \mathbb{C}^* = T_{N_2}.$$

Remark 3.2.8. One can show that $\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$ is proper if and only if $\varphi^{-1}(|\Sigma_2|) = |\Sigma_1|$. See [CLS11, Thm 3.4.11].

Example 3.2.9. Suppose φ is the identity and Σ_1 is a refinement of Σ_2, i.e. $|\Sigma_1| = |\Sigma_2|$ and every cone $\sigma_1 \in \Sigma_1$ is contained in some cone $\sigma_2 \in \Sigma_2$. Then the corresponding map $X_{\Sigma_1} \to X_{\Sigma_2}$ is proper and birational.

3.3 The orbit-cone correspondence.

A cone $\sigma \in \Sigma$ defines a distinguished point $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$: γ_σ is given by the semigroup morphism:

$$m \in S_\sigma \mapsto \begin{cases} 1, & m \in \sigma^\perp \cap M \\ 0, & \text{otherwise} \end{cases}$$

This is a fixed point for the T_N action if and only if $\dim \sigma = \dim N_\mathbb{R}$.

Theorem 3.3.1 ([CLS11, Theorem 3.2.6 and Prop. 3.2.7])**.** *There is a bijective correspondence*

$$\{\sigma \in \Sigma\} \longleftrightarrow \{T_N\text{-orbits} \subseteq X_\Sigma\}$$
$$\sigma \longleftrightarrow O(\sigma) := T_N \cdot \gamma_\sigma$$

having the following properties:

1. $\dim_\mathbb{R} \sigma + \dim_\mathbb{C} O(\sigma) = \dim_\mathbb{R} N_\mathbb{R}$

2. The affine open set U_σ decomposes into orbits as

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$$

3. $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$V(\tau) := \overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma).$$

3.4 Divisors and the homogeneous coordinate ring.

Let X_Σ be a toric variety associated to the fan Σ. A one-dimensional cone $\rho \in \Sigma(1)$ gives a torus-invariant divisor $D_\rho = V(\rho)$ under the orbit-cone correspondence and every torus-invariant divisor is a sum of these. Denoting the latter group by $Div_T(X_\Sigma)$, we have an identification

$$\mathbb{Z}^{\Sigma(1)} \cong Div_T(X_\Sigma).$$

Since ρ is a one-dimensional rational cone, there is a unique smallest lattice generator of ρ, i.e. an element $u_\rho \in N$ such that $\rho = \mathbb{R}_{\geq 0} u_\rho$.

Proposition 3.4.1 ([CLS11, Thm 4.1.3])**.** *There is an exact sequence*

$$M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

where $\mathrm{Cl}(X_\Sigma)$ denotes the class group. The first morphism maps $m \in M$ to the divisor

$$div(t^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

of the rational function t^m. The second is the natural quotient map $\mathbb{Z}^{\Sigma(1)} \cong Div_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$. If X_Σ has no torus factors, i.e. it is not of the form $X_\Sigma \cong X_{\Sigma'} \times T^k$, then there is a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0.$$

The global sections of torus-invariant divisors are described by polyhedra as follows ([CLS11, Prop. 4.3.3]): If $D = \sum_\rho a_\rho D_\rho$ is a torus-invariant divisor on X_Σ, then

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot t^m,$$

where

$$P_D := \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho\}.$$

Now suppose X_Σ is a smooth toric variety without torus factors. We want to describe X_Σ by a graded ring S_Σ, generalizing the homogeneous coordinate description of projective space. Tensoring the exact sequence of the class group with $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ gives the exact sequence

$$1 \longrightarrow G_\Sigma \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T_N \longrightarrow 1$$

where $G_\Sigma = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ is the character group of $\mathrm{Cl}(X_\Sigma)$. This is a reductive group isomorphic to the product of a torus and a finite group. We can describe G_Σ concretely as

$$G_\Sigma = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_{\rho \in \Sigma(1)} t_\rho^{\langle m, u_\rho \rangle} = 1 \text{ for all } m \in M\}.$$

Definition 3.4.2. The *homogeneous coordinate ring* of X_Σ is

$$S_\Sigma = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)] = \mathcal{O}(\mathbb{C}^{\Sigma(1)}).$$

The ring S_Σ is graded by $\mathrm{Cl}(X)$:

$$\deg(x_\rho) = [D_\rho]$$

This gives an action of G_Σ by duality, which is just the restriction of the natural scaling action of $(\mathbb{C}^*)^{\Sigma(1)}$. The corresponding eigenspaces are the graded components of S_Σ:

$$S_\Sigma = \bigoplus_{\beta \in \mathrm{Cl}(X)} S_\beta.$$

We want to describe X_Σ as a suitable quotient of $\mathrm{Spec}(S_\Sigma) = \mathbb{C}^{\Sigma(1)}$ by G_Σ. For this to work, we first have to throw out some badly behaved G_Σ-orbits.

Definition 3.4.3. For $\sigma \in \Sigma$ let $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \in S_\Sigma$. The *irrelevant ideal* is

$$B_\Sigma = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle.$$

The corresponding zero set $Z_\Sigma = V(B_\Sigma)$ is the *irrelevant locus.*

The variety $\mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma$ is again toric. Its fan can be described as follows: For $\sigma \in \Sigma$ let

$$\tilde{\sigma} = \mathrm{pos}(e_\rho \mid \rho \in \sigma(1)) \subseteq \mathbb{R}^{\Sigma(1)}.$$

The collection of all $\tilde{\sigma}$ constitute the fan $\tilde{\Sigma}$ of $\mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma$. The lattice morphism

$$\overline{\pi} : \mathbb{Z}^{\Sigma(1)} \to N, \quad e_\rho \mapsto u_\rho$$

is obviously compatible with the fans. Hence we get a toric morphism

$$\pi : \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma \to X_\Sigma.$$

Theorem 3.4.4 ([CLS11, Thm. 5.1.11])**.** *Let X_Σ be a smooth toric variety without torus factors. The map π describes X_Σ as the geometric quotient*

$$X_\Sigma = \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma // G_\Sigma$$

Remark 3.4.5. We refer to [CLS11, Section 5.0] for background on geometric invariant theory and geometric quotients. Let us point out some consequences of this result:

1. The G_Σ-orbits on $\mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma$ are closed and the set of closed points of X_Σ is the orbit space.

2. For an affine open subset $U = \mathrm{Spec}(R) \subseteq X_\Sigma$ we have $\pi^{-1}(U) = \mathrm{Spec}(\tilde{S})$, where $\tilde{S}$ is a localization of S_Σ with an induced G_Σ-action. That π is a geometric quotient implies that $R = \tilde{S}^{G_\Sigma}$, i.e. R is the subring of G_Σ-invariants.

Let us specialize the above remark to an affine open $U_\sigma \subseteq X_\Sigma$ given by a cone $\sigma \in \Sigma$. For the inverse image we have

$$\pi^{-1}(U_\sigma) = U_{\tilde{\sigma}} = \mathrm{Spec}(\mathbb{C}[\tilde{\sigma}^\vee \cap \mathbb{Z}^{\Sigma(1)}]),$$

where $\tilde{\sigma} = \mathrm{pos}(e_\rho \mid \rho \in \sigma(1))$. The coordinate ring is then

$$\mathbb{C}[\tilde{\sigma}^\vee \cap M] = \mathbb{C}\left[\prod_\rho x_\rho^{a_\rho} \mid a_\rho \geq 0 \text{ for } \rho \in \sigma(1)\right] := S_{x^{\hat{\sigma}}},$$

i.e. we invert every variable x_ρ for $\rho \notin \sigma(1)$. Hence we get

$$\pi^{-1}(U_\sigma) = \mathrm{Spec}(S_{x^{\hat{\sigma}}}).$$

The map on coordinate rings is given by homogenization:

$$\begin{aligned} \pi^* : \mathbb{C}[\sigma^\vee \cap M] &\longrightarrow S_{x^{\hat{\sigma}}} \\ \pi^*(t^m) &= \prod_\rho x_\rho^{\langle m, u_\rho \rangle} \end{aligned}$$

Its image is the space of G_Σ-invariants $S_{x^{\hat{\sigma}}}^{G_\Sigma}$. This gives the isomorphism

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \cong \mathrm{Spec}(S_{x^{\hat{\sigma}}}^{G_\Sigma}).$$

For top-dimensional cones $\sigma \in \Sigma(\dim N)$, we can describe the above isomorphism by dehomogenization, i.e. setting some of the variables x_ρ to 1:

Proposition 3.4.6. *Let $\sigma \in \Sigma(\dim N)$ be a top-dimensional cone and*

$$Y_\sigma = \{x_\rho \in \mathbb{C}^{\Sigma(1)} \mid x_\rho = 1 \textit{ for } \rho \notin \sigma(1)\}.$$

There is a G_Σ-equivariant isomorphism $p^{-1}(U_\sigma) \cong Y_\sigma \times G_\Sigma$, fitting into the commutative diagram

$$\begin{tikzcd}
\{1\} \times Y_\sigma \arrow[r, equal] \arrow[d, hook] & Y_\sigma \arrow[d, hook] \arrow[dr] & \\
G_\Sigma \times Y_\sigma \arrow[r] & p^{-1}(U_\sigma) \arrow[r] & U_\sigma
\end{tikzcd}$$

and inducing an isomorphism $Y_\sigma \cong U_\sigma$.

Proof. From the above discussion, it follows that

$$\pi_\Sigma^{-1}(U_\sigma) = \{x \in \mathbb{C}^{\Sigma(1)} \mid x_\rho \neq 0 \text{ for } \rho \notin \sigma(1)\} \subseteq \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma,$$

and the middle vertical arrow is well-defined. The natural map

$$\phi : G_\Sigma \times Y_\sigma \to \pi_\Sigma^{-1}(U_\sigma), \quad \phi(t, x) = t \cdot x,$$

is clearly G_Σ-equivariant. Its inverse is constructed as follows: Since σ is smooth, the ray generators $(u_\rho \mid \rho \in \sigma(1))$ constitute a $\mathbb{Z}$-basis of N. Let $(m_\rho \mid \rho \in \sigma(1))$ be the dual basis of $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$.

For $x \in \pi_\Sigma^{-1}(U_\sigma)$, let

$$t(x)_\rho = \begin{cases} \prod_{\tilde\rho \notin \sigma(1)} x_{\tilde\rho}^{-\langle m_\rho, u_{\tilde\rho} \rangle}, & \rho \in \sigma(1) \\ x_\rho, & \rho \notin \sigma(1) \end{cases}$$

Then $t(x)^{-1}x \in Y_\sigma$ by construction. For $m = m_\rho$, we have

$$\prod_{\tilde\rho \in \Sigma(1)} t(x)_{\tilde\rho}^{\langle m_\rho, u_{\tilde\rho} \rangle} = t(x)_\rho \prod_{\tilde\rho \notin \sigma(1)} x_\rho^{\langle m_\rho, u_\rho \rangle} = 1$$

and since the m_ρ form a basis of M, we get $t(x) \in G_\Sigma$. Hence the map

$$\psi : \pi_\Sigma^{-1}(U_\sigma) \to G_\Sigma \times Y_\sigma, \quad \psi(x) = (t(x), t(x)^{-1}x).$$

is well-defined and clearly inverse to ϕ. □

Example 3.4.7. The classical example is the homogeneous coordinate description of projective space P^n. Let $e_i \in \mathbb{Z}^n$ be the standard basis vectors and $e_0 = -\sum_i e_i$. The fan Σ of P^n then consists of the cones

$$\sigma_I = \mathrm{pos}(e_i \mid i \in I),$$

where $I \subsetneq \{0, 1, \ldots, n\}$ varies over the proper subsets of $\{0, 1, \ldots, n\}$. Then $\mathbb{C}^{\Sigma(1)} \cong \mathbb{C}^{n+1}$ with coordinates $(x_0, \ldots, x_n)$ and $Z_\Sigma = \{0\}$. The map $u = \sum_{i=0}^n u_i e_i \mapsto \sum_{i=0}^n u_i$ induces an isomorphism $\mathrm{Cl}(P^n) \cong \mathbb{Z}$, such that $\deg(x_i) = 1$ for $i = 0, \ldots, n$. The corresponding character group is

$$G_\Sigma = \{t \in (\mathbb{C}^*)^{\Sigma(1)} \mid t_0^{-1} t_i = 1 \text{ for } i = 1, \ldots, n\} \cong \mathbb{C}^*(1, \ldots, 1).$$

Then we recover the usual isomorphism $P^n \cong \mathbb{C}^{n+1} \backslash \{0\} // \mathbb{C}^*$.

Example 3.4.8. Let $B^n = Bl_0 P^n$ be the blowup of P^n in $x = 0$. We will see below that B^n is a smooth toric variety and its fan can be described as follows: Let $e_z = \sum_{i=1}^n e_i$ and

$$\Sigma(1) = \{\mathbb{R}_{\geq 0} e_z\} \cup \{\mathbb{R}_{\geq 0} e_i \mid i = 0, \dots, n\}.$$

Then $\sigma_I = \mathrm{pos}(e_i \mid i \in I)$ is a cone of Σ iff $I \subseteq \{z, 0, \dots, n\}$ satisfies $\{1, \dots, n\} \nsubseteq I$ and $\{z, 0\} \nsubseteq I$. We can identify $\mathrm{Cl}(B^n) \cong \mathbb{Z}^2$ through the map

$$\mathbb{Z}^{n+2} \to \mathbb{Z}^2, \quad u \mapsto (\sum_{i=1}^n u_i - u_z, \sum_{i=0}^n u_i).$$

The corresponding variables $(x_z, x_0, \dots, x_n)$ have degrees $\deg(x_i) = (1,1), \deg(x_0) = (0,1)$ and $\deg(x_z) = (-1, 0)$. Similarly we get

$$G_\Sigma = \{(t_z, t_0, \frac{t_0}{t_z}, \dots, \frac{t_0}{t_z}) \in (\mathbb{C}^*)^{n+2} \mid t_z, t_0 \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^2$$

and

$$Z_\Sigma = \bigcap_{i=1}^n V(x_i x_z) \cap V(x_i x_0).$$

Thus the blowup variety has the quotient description $B^n = \mathbb{C}^{n+2} \backslash Z_\Sigma // (\mathbb{C}^*)^2$.

There is a general correspondence between graded S_Σ-modules and quasi-coherent sheaves on X_Σ. We want to explain this in the case of the canonical sheaf ω_Σ, the sheaf of top-degree differential forms. Let $(e_1, \dots, e_n)$ be a basis of M and $I = \{\rho_1, \dots, \rho_n\} \subseteq \Sigma(1)$ an n-element subset. Let $\epsilon_I = \det(\langle e_i, u_{\rho_j} \rangle_{ij})$ and set

$$\Omega_\Sigma = \sum_I \epsilon_I \left(\prod_{\rho \notin I} x_\rho \right) dx_{\rho_1} \wedge \ldots \wedge dx_{\rho_n}.$$

This is an element of the S_Σ-module

$$\bigwedge^n \Omega^1_{S_\Sigma} \cong \Gamma(\mathbb{C}^{\Sigma(1)}, \omega^n_{\mathbb{C}^{\Sigma(1)}}),$$

where $\Omega^1_{S_\Sigma}$ is the module of Kähler differentials over S_Σ. Note that Ω_{X_Σ} is independent of the above choices up to a sign, which can be fixed by choosing an orientation of M and requiring that $(e_1, \dots, e_n)$ is an oriented basis

The group G_Σ acts on $\bigwedge \Omega^1_{S_\Sigma}$ by pullback. The action of $t^m \in G_\Sigma \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ is given by

$$t^m \cdot \Omega_\Sigma = t^{\langle m, \sum_\rho u_\rho \rangle} \Omega_\Sigma.$$

Hence Ω_Σ has degree

$$\beta = \left[\sum_\rho D_\rho \right] \in \mathrm{Cl}(X_\Sigma).$$

If $F, H \in S_\Sigma$ are polynomials, such that $\deg F - \deg H = -\beta$, then the meromorphic n-form

$$\frac{F(x)}{H(x)}\Omega_\Sigma$$

is G_Σ-invariant and descends to a global meromorphic section of ω_Σ. Conversely, let $f, h \in \mathcal{O}(T_N)$ be Laurent polynomials and consider the section

$$\alpha = \frac{f(t)}{h(t)}\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n} \in \Gamma(T_N, \omega_{T_N}),$$

where $t_1, \ldots, t_n$ are generators of $\mathcal{O}(T_N)$. Pulling back along the quotient map $\pi : \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma \to X_\Sigma$ gives a meromorphic form $\pi^*\alpha$ on $\mathbb{C}^{\Sigma(1)}$, which we can describe as follows.

Proposition 3.4.9. *Let $F = \pi^* f$ and $H = \pi^* h \prod_\rho x_\rho$. Then the pullback $\pi^*\alpha$ is given by*

$$\pi^*\alpha = \frac{F(x)}{H(x)}\Omega_\Sigma.$$

Proof. From $\pi^*(t^m) = \prod_\rho x_\rho^{\langle m, u_\rho \rangle}$ it follows that

$$\pi^*\left(\frac{dt_1}{t_1}\right) = \sum_\rho \langle e_i, u_\rho \rangle \frac{dx_\rho}{x_\rho}.$$

Taking the wedge product and multiplying by $\prod_\rho x_\rho$ gives

$$\left(\prod_\rho x_\rho\right)\pi^*\left(\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}\right) = \Omega_\Sigma$$

and the above formula follows. $\square$

Over the maximal cone $\sigma = \mathrm{pos}(u_i \mid i \in I_0) = \mathrm{pos}(u_{i_1} \ldots, u_{i_n})$, we formally set $x_\rho = 1$ for $\rho \notin \sigma(1)$ and get the section

$$\alpha\big|_{U_\sigma} = \epsilon_\sigma \frac{\tilde{f}(x)}{\tilde{g}(x)} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_n} \in \Gamma(U_\sigma, \omega_\Sigma),$$

where $\epsilon_\sigma = \epsilon_{I_0}$ and $\tilde{f} = F|_{Y_\sigma}$ and $\tilde{h} = H|_{Y_\sigma}$ are the dehomogenizations of F and H.

3.5 Lattice polytopes.

Now let $P \subseteq M_\mathbb{R}$ be a lattice polytope, i.e. it is the convex hull of finitely many lattice points. Suppose also that P is of full dimension $n = \dim M$. Then it can be described by the facet presentation

$$P = \{m \in M \mid \langle m, u_F \rangle \geq -a_F, F \in P(n-1)\},$$

where $a_F \in \mathbb{Z}$ and u_F is the minimal lattice generator of the cone ρ_F consisting of inward pointing normal vectors of F.

We can construct the *normal fan* of P as follows: Let $v \in P$ be a vertex and C_v the cone generated by $P \cap M - v$. Its dual cone $\sigma_v = C_v^\vee$ is again rational and strongly convex. In terms of the facet presentation above, we have

$$\sigma_v = \mathrm{pos}(u_F \mid F \in P(n-1), v \in F).$$

More generally, we can define for any face $Q \in P(k)$ the cone

$$\sigma_Q = \mathrm{pos}(u_F \mid F \in P(n-1), Q \subseteq F).$$

Proposition 3.5.1 ([CLS11, Prop. 2.3.7 and Prop. 3.1.6].)**.** *The cones σ_Q constitute a complete fan Σ_P in $N_\mathbb{R}$ and define a complete toric variety $X_P := X_{\Sigma_P}$. A vector $u \in N_\mathbb{R}$ defines the face $F_u P = Q$ if and only if $u \in \mathrm{relint}(\sigma_Q)$. This defines an inclusion reversing bijection between Σ_P and the set of faces of P. If P is not full-dimensional, then the same construction will give a generalized fan.*

Remark 3.5.2. The orbit-cone correspondence takes the following form: The cones $\sigma \in \Sigma_P$ correspond to faces $Q \subseteq P$, hence every Q gives a torus orbit $O(Q) \subseteq \Sigma_P$ and its closure $V(Q)$. The latter is a closed toric subvariety and hence again given by a complete fan, which we can describe as follows: By translating Q by one of its vertices we can assume that $0 \in Q$. Let then M_Q be the linear span of Q such that $Q \subseteq M_Q$ becomes a full-dimensional lattice polytope which has a normal fan Σ_Q. Then we have $V(Q) \cong X_Q$. See [CLS11, Prop 3.2.9].

Remark 3.5.3. Identifying rays $\rho \in \Sigma_P(1)$ with facets F of P, we get the divisor $D_P = \sum_F a_F D_F$ canonically attached to X_P. One can show that D_P is ample and there is a bijective correspondence

$$P \longleftrightarrow (X_\Sigma, D)$$

between full dimensional lattice polytopes $P \subseteq M_\mathbb{R}$ and complete toric varieties X_Σ with fan $\Sigma \subseteq N_\mathbb{R}$ together with a distinguished torus-invariant ample divisor D. See [CLS11, Thm. 6.2.1].

The following proposition will be needed later.

Proposition 3.5.4. *Suppose $P \subseteq M$ is a (not necessarily full-dimensional) lattice polytope. An n-dimensional cone $\sigma = \mathrm{pos}(u_1, \ldots, u_n)$ is contained in a maximal cone $\tilde{\sigma} \in \Sigma_P(n)$ if and only if there is a unique vertex $m_\sigma \in P(f)$, such that*

$$\langle m_\sigma, u_i \rangle = \min_{m \in P} \langle m, u_i \rangle$$

for all $i = 1, \ldots, n$.

Proof. Let $\tilde{\sigma} \in \Sigma_P(n)$, correspond to the vertex $m_0 \in P(0)$. The cone

$$\sigma = \mathrm{pos}(u_1, \ldots, u_n)$$

is contained in $\tilde{\sigma}$ if and only if every weight vector $w = \sum_{i=1}^{n} \lambda_i u_i \in \mathrm{Int}(\sigma)$ defines the face $F_w P = \{m_0\}$. This means that

$$\langle m_0, w \rangle < \langle m, w \rangle, \text{ for all } m \in P \backslash \{m_0\}.$$

Varying λ_i, it is easy to see that this is only possible if

$$\langle m_0, u_i \rangle = \min_{m \in P} \langle m, u_i \rangle,$$

for all $i = 1, \ldots, n$. Conversely, suppose m_0 minimizes $\langle m, u_i \rangle$ for all i and thus for all $w \in \sigma$. Suppose there is another $\tilde{m} \in P$, such that $\langle \tilde{m}, w \rangle$ is minimal. Then $\langle \tilde{m} - m_0, w \rangle = 0$ for all $w \in \sigma$, which implies $\tilde{m} = m_0$ since σ is full-dimensional. □

For two lattice polytopes $P_1, P_2 \subseteq M_{\mathbb{R}}$, let

$$Q = P_1 + P_2 = \{m_1 + m_2 \mid m_1 \in P_1, m_2 \in P_2\}$$

be their Minkowski sum. This is clearly a lattice polytope again.

Proposition 3.5.5 ([Zie95, Prop. 7.12])**.** *The normal fan Σ_Q of Q is the coarsest common refinement of the normal fans $\Sigma_{P_1}, \Sigma_{P_2}$.*

Note that the normal fan of a polytope P is well-defined even if P is not a lattice polytope. This gives a convenient notion of equivalence.

Definition 3.5.6. Two polytopes $P_1, P_2 \subseteq M_{\mathbb{R}}$ are called *normally* equivalent, if they have the same normal fan.

Example 3.5.7. For a polytope $P \subseteq M_{\mathbb{R}}$ and $r > 0$ the scaled polytopes rP are clearly all normally equivalent to each other. More generally let $P_1, \ldots, P_k \subseteq M_{\mathbb{R}}$ be polytopes and $r \in (0, \infty)^k$. Then the normal fan of

$$P(r) = \sum_{i=1}^{k} r_i P_i$$

is the least common refinements of the normal fans Σ_{P_i} by Prop. 3.5.5. Hence the normal equivalence class of $P(r)$ is independent of r.

3.6 Real and real-positive locus.

Let $\sigma \in \Sigma$ be a cone in the fan defining X_Σ. The complex points of the corresponding affine toric variety are given by

$$U_\sigma(\mathbb{C}) = \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{C}).$$

Restricting the image to $\mathbb{R}$ gives the locus $U_\sigma(\mathbb{R})$. These glue together to give the real locus $X_\Sigma(\mathbb{R})$. Similarly, restricting to $\mathbb{R}_{\geq 0}$ gives the real-positive locus $X_\Sigma(\mathbb{R}_{\geq 0})$. A toric morphism $X_\Sigma \to X_{\tilde{\Sigma}}$ induces maps $X_\Sigma(\mathbb{R}) \to X_{\tilde{\Sigma}}(\mathbb{R})$ and $X_\Sigma(\mathbb{R}_{\geq 0}) \to X_{\tilde{\Sigma}}(\mathbb{R}_{\geq 0})$.

Example 3.6.1. Suppose X_Σ is a projective toric variety associated to the polytope P, such that the divisor D_P is very ample. Its sections

$$t^{m_i} \in \Gamma(X_\Sigma, \mathcal{O}_{X_P}(D_P)), \quad m_i \in P \cap M$$

furnish a projective embedding

$$X_\Sigma \to \mathbb{P}^s, \quad x \mapsto [t^{m_0}(x) : \ldots : t^{m_s}(x)].$$

The (algebraic) moment is defined as

$$\begin{aligned} f : X_\Sigma &\to M_\mathbb{R} \\ f(x) &= \frac{\sum_{m\in P\cap M} |t^m(x)| m}{\sum_{m \in P\cap M} |t^m(x)|}. \end{aligned}$$

By [CLS11, Thm. 12.2.5], this induces a homeomorphism

$$f : X_\Sigma(\mathbb{R}_{\geq 0}) \xrightarrow{\sim} P,$$

which identifies a facet $Q \subseteq P$ with $V(Q) \cap X_\Sigma(\mathbb{R}_{\geq 0})$.

3.7 Star subdivision of fans.

There is a standard construction to refine a given fan Σ. Let $\nu \in \Sigma \cap N$ be a primitive element, i.e. such that ν is the lattice generator of $\mathrm{pos}(\nu)$. For $\sigma \in \Sigma$ with $\nu \in \sigma$ let

$$\Sigma_\sigma(\nu) = \{\mathrm{pos}(\tau, \nu) \mid \{\nu\} \cup \tau \subseteq \sigma, \nu \notin \tau\}.$$

The *star subdivision* of Σ with respect to ν is the fan

$$\Sigma^*(\nu) = \{\sigma \in \Sigma \mid \nu \notin \sigma\} \cup \bigcup_{\nu \in \sigma} \Sigma_\sigma(\nu).$$

The identity map $N \to N$ is compatible with the fans $(\Sigma^*(\nu), \Sigma)$ and induces a toric morphism

$$\pi : X_{\Sigma^*(\nu)} \to X_\Sigma,$$

which is proper and birational.

We are interested in the following special case of this construction. Let X_Σ be a smooth toric variety associated to the fan Σ. From Prop. 3.2.5 we know that every cone $\sigma \in \Sigma$ is smooth, i.e. can be generated by part of a $\mathbb{Z}$-basis of N. For a cone $\tau \in \Sigma$, the closure $V(\tau) = \overline{O}(\tau)$ is a smooth toric subvariety. Let Σ_τ be the star subdivision of Σ with respect to the vector

$$\nu_\tau = \sum_{\rho \in \tau(1)} u_\rho$$

Proposition 3.7.1 ([Oda88, Prop. 1.26]). *The map $\pi : X_{\Sigma_\tau} \to X_\Sigma$ is the blowup of X_Σ with center $V(\tau)$.*

3.8 Toric wonderful models.

In this section, we want to describe certain compactifications of the torus T^{n-1} given by iteratively blowing up coordinate subspaces in the projective compactification $T^{n-1} \hookrightarrow P^{n-1}$. These are special cases of the wonderful model compactifications of [DCP95].

Let us first give a more symmetric description of the fan in Example 3.4.7. Suppose E is a finite set with n elements and let P^E be the projective space of dimension $n-1$, where we label the homogeneous coordinates by elements of E. Let

$$N_E = \mathbb{Z}^E / \mathbb{Z}\left(\sum_{i \in E} e^i\right) \cong \mathbb{Z}^{n-1}$$

and

$$M_E = \left\{ m \in \mathbb{Z}^E \mid \sum_{i \in E} m_i = 0 \right\}$$

the dual lattice. The fan Σ_E of P^E is given by the cones

$$\tau_I := \mathrm{pos}([e^i] \mid i \in I)$$

for all $I \subsetneq E$. Every such proper subset $I \subsetneq E$ then gives the linear subspace

$$L_I = \{[\alpha_j] \mid \alpha_i = 0 \text{ for } i \in I\} \cong P^{I^c},$$

which is the orbit closure associated to the cone τ_I.

Consider a set of subsets $B \subseteq 2^E$ satisfying the following conditions.

1. $E \notin B$
2. $\{i\} \notin B$ for all $i \in E$.
3. $I_1, I_2 \in B, I_1 \cap I_2 \neq \emptyset \Rightarrow I_1 \cup I_2 \in B$ or $I_1 \cup I_2 = E$.

The iterated blowup

$$\pi_B : P^B \to P^E$$

is defined by inductively blowing up the elements of

$$\mathcal{L}_B = \{L_I \mid I \in B\},$$

in order of increasing dimension. More precisely, let $B = \{I_1 < \ldots < I_m\}$ be linearly ordered such that $j \geq k$ implies that $I_j \subseteq I_k$. We then define the sequence of blowups by $P_0 = P^E$ and $P_k = Bl_{\tilde{L}_{I_k}} P_{k-1}$, where $\tilde{L}_k$ is the strict transform of L_k in P_{k-1}. The results of the last section show that this is again a smooth projective toric variety. Let Σ_k be the fan of P_k. Then we have that $\Sigma_k = St_{\tau_k}\Sigma_{k-1}$ is the star subdivision with respect to the cone

$$\tau_k = \mathrm{pos}\{[e^i] \mid i \in I_k\}.$$

Blowing up in order of increasing dimension ensures that this is well-defined, i.e. that τ_k is indeed a cone of Σ_{k-1}. The strict transform of L_{I_k} in P_{k-1} is just the orbit closure $V(\tau_k)$ of $\tau_k \in \Sigma_{k-1}$. Let $P^B = P_m$ be the last blowup and $\Sigma_B = \Sigma_m$ its fan.

To fully describe the fan, we will use the combinatorial approach to wonderful models developed in [FK04]. Note that any fan $(\Sigma, \preceq)$ with its face relation is a meet semi-lattice, i.e. every collection $\sigma_1, \dots, \sigma_k \in \Sigma$ has the greatest lower bound

$$\bigwedge_i \sigma_i = \bigcap_i \sigma_i \in \Sigma.$$

The minimal element of Σ is the trivial cone $\{0\}$. For $\Sigma = \Sigma_E$, there is an obvious poset isomorphism

$$(\Sigma_E, \preceq) \cong (2^E \backslash \{E\}, \subseteq),$$

identifying a subset $I \subsetneq E$ with the cone τ_I.

Now let $(\mathcal{L}, \preceq)$ be any finite meet-semilattice with least element $\hat{0}$. For a subset $\mathcal{G} \subseteq \mathcal{L}$, and $X \in \mathcal{L}$, let

$$\mathcal{G}^{\preceq X} := \{G \in \mathcal{G} \mid G \preceq X\}$$

and

$$[\hat{0}, X] = \{Y \in \mathcal{L} \mid \hat{0} \preceq Y \preceq X\}.$$

Definition 3.8.1. Let $(\mathcal{L}, \preceq)$ be a finite meet-semilattice. A subset $\mathcal{G} \subseteq \mathcal{L} \backslash \{\hat{0}\}$ is called a *building set*, if the following holds for all $X \in \mathcal{L}$: Let

$$\max \mathcal{G}^{\preceq X} = \{G_1, \dots, G_k\}$$

be the maximal elements of $\mathcal{G}^{\preceq X}$. Then there is an order isomorphism

$$[\hat{0}, X] \cong \prod_{i=1}^{k} [\hat{0}, G_i].$$

Remark 3.8.2. Note that every building set must contain all product-irreducible elements, i.e. all elements $X \in \mathcal{L}$, such that $[\hat{0}, X]$ can not be expressed as a non-trivial product. This includes in particular all atoms of $\mathcal{L}$, i.e. all elements $a \in \mathcal{L} \backslash \hat{0}$, such that $[\hat{0}, a] = \{\hat{0}, a\}$.

Example 3.8.3. Suppose $\mathcal{L} = 2^E \backslash \{E\}$ with the subset relation. Then a subset $\mathcal{G} \subseteq \mathcal{L} \backslash \emptyset$ is a building set if for all $I \subsetneq E$, the maximal elements

$$\max \mathcal{G}^{\subseteq I} = \{G_1, \dots, G_k\}$$

form a partition $I = \coprod_{i=1}^k G_i$. It is easy to check that this is equivalent to the condition that $\mathcal{G}$ contains all singleton subsets and for all $I_1, I_2 \in \mathcal{G}$:

$$I_1 \cap I_2 \neq \emptyset \Rightarrow I_1 \cup I_2 \in \mathcal{G} \text{ or } I_1 \cup I_2 = E.$$

The set $\tilde{\mathcal{G}} = \mathcal{G} \cup \{E\}$ is then a building set in 2^E.

To describe the face structure of Σ_B, we will also need the notion of nested sets.

Definition 3.8.4. Let $\mathcal{G} \subseteq \mathcal{L} \backslash \hat{0}$ be a building set in a finite meet-semilattice. A subset $\mathcal{N} \subseteq \mathcal{G}$ is called nested if for all pairwise non-comparable elements $N_1, \ldots, N_k \in \mathcal{N}$ with $k \geq 2$, the join $\bigvee_{i=1}^{k} N_i \in \mathcal{L}$ exists in $\mathcal{L}$ but is not in $\mathcal{G}$.

Example 3.8.5. Suppose $\mathcal{G} \subseteq 2^E \backslash \{E\}$ is a building set. Then $\mathcal{I} \subseteq \mathcal{G}$ is a nested set if and only if:

1. For all I_1, I_2, either $I_1 \cap I_2 = \emptyset$ or $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$.
2. If $I_1, \ldots, I_k \in \mathcal{I}$ are pairwise disjoint and $k \geq 2$, then

$$\bigcup_{j=1}^{k} I_j \notin \mathcal{G} \cup \{E\}.$$

It follows from [FK04, Prop. 2.8] that all maximal nested sets are generated by the following construction: Let $E = \{i_1 < \ldots < i_n\}$ be a total ordering of E. Set $J_k = \{i_1, \ldots, i_k\}$ and $\mathcal{I}_k = \max \mathcal{G}^{\subseteq J_k}$. The union $\mathcal{I} = \bigcup_{k=1}^{n} \mathcal{I}_k$ is then a maximal nested set.

The nested sets of a building set $\mathcal{G}$ are partially ordered by inclusion. We denote the corresponding poset by $\mathcal{N}(G)$. We can now state the results of [FK04, Theorem 4.10].

Theorem 3.8.6. *Let Σ be the fan of a toric variety X_Σ and $\mathcal{G} \subseteq (\Sigma, \preceq)$ be a building set in its face semilattice. Suppose $\mathcal{G} = \{\tau_1 < \ldots < \tau_k\}$ is linearly ordered, such that $\tau_i < \tau_j$ implies $\tau_j \preceq \tau_i$. Let $\Sigma_{\mathcal{G}}$ be the fan obtained by subdividing Σ along the G_i in increasing order. Then there is an isomorphism of semilattices*

$$(\Sigma_{\mathcal{G}}, \preceq) \cong (\mathcal{N}(G), \subseteq),$$

identifying a nested set $\mathcal{N} = \{\tau_{i_1}, \ldots, \tau_{i_s}\} \subseteq G$ with the cone

$$\tau_{\mathcal{N}} = \mathrm{pos}(\nu_\tau \mid \tau \in N),$$

where $\nu_\tau = \sum_{\rho \in \tau(1)} u_\rho$.

Remark 3.8.7. The rays of $\Sigma_{\mathcal{G}}$ are the one-element nested sets, which correspond to the elements of $\mathcal{G}$. This gives a natural bijection $\Sigma_{\mathcal{G}}(1) \cong \mathcal{G}$.

If X_Σ is smooth, then so is its iterated blow-up $X_{\mathcal{G}} = X_{\Sigma_{\mathcal{G}}}$. Its homogeneous coordinate description takes the following form.

Proposition 3.8.8. *Suppose X_Σ is a smooth toric variety without torus factors and $\mathcal{G} \subseteq \Sigma$ a building set in its fan. Denote by $\mathcal{G}^* \subseteq \mathcal{G}$ the subset of cones which are not rays and let $\pi_{\mathcal{G}} : X_{\mathcal{G}} \to X_\Sigma$ be the toric map obtained by iteratively blowing up elements of $\mathcal{G}^*$ as above.*

1. *There is a natural isomorphisms $\mathrm{Cl}(X_{\mathcal{G}}) \cong \mathrm{Cl}(X_\Sigma) \oplus \mathbb{Z}^{\mathcal{G}^*}$ The corresponding character group is $G_{\mathcal{G}} := G_{\Sigma_{\mathcal{G}}} \cong G_\Sigma \times (\mathbb{C}^*)^{\mathcal{G}^*}$.*

2. The exceptional locus $Z_{\mathcal{G}} := Z_{\Sigma_{\mathcal{G}}} \subseteq \mathbb{C}^{\mathcal{G}}$ *is given by*

$$Z_{\mathcal{G}} = \bigcap_{\mathcal{N} \subseteq \mathcal{G} \text{ nested}} V(\prod_{\tau \notin \mathcal{N}} x_\tau).$$

3. The variety $X_{\mathcal{G}}$ *can be described as the geometric quotient*

$$X_{\mathcal{G}} \cong (\mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}) // G_{\mathcal{G}}.$$

Proof. The class groups fit into the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^{\mathcal{G}^*} \arrow[r] \arrow[d] & \mathbb{Z}^{\mathcal{G}^*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathbb{Z}^{\Sigma(1)} \bigoplus \mathbb{Z}^{\mathcal{G}^*} \arrow[r] \arrow[d] & \mathrm{Cl}(X_{\mathcal{G}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathbb{Z}^{\Sigma(1)} \arrow[r] \arrow[d] & \mathrm{Cl}(X_\Sigma) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The pullback $\pi_{\mathcal{G}}^* : \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(X_{\mathcal{G}})$ induces a splitting

$$\mathrm{Cl}(X_{\mathcal{G}}) \cong \mathrm{Cl}(X_\Sigma) \oplus \mathbb{Z}^{\mathcal{G}^*}.$$

Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ gives the isomorphism $G_{\mathcal{G}} \cong G_\Sigma \times (\mathbb{C}^*)^{\mathcal{G}^*}$.

The description of the exceptional locus and the geometric quotient is now immediate from Thm. 3.4.4 and Thm. 3.8.6. $\square$

To our original set $B \subseteq 2^E$, we associate the set

$$\mathcal{G}_B = B \cup \{\{i\} \mid i \in E\}.$$

This is a building set by Exam. 3.8.3. To $I \subseteq E$, we associate the vector

$$e^I = \sum_{i \in I} e^i.$$

Applying the above theorem to $\mathcal{G}_B$ then gives:

Corollary 3.8.9. P^B *is a smooth, projective variety, independent of the chosen blowup-order. Its fan* Σ_B *consists of the cones*

$$\sigma_{\mathcal{I}} = \mathrm{pos}([e^I] \mid I \in \mathcal{I}),$$

where $\mathcal{I} \subseteq \mathcal{G}_B$ *ranges over the nested sets with respect to* $\mathcal{G}_B$.

In particular we have $\Sigma_B(1) \cong B \cup E$. The map $\pi_B : P^B \to P^E$ fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{\Sigma_B(1)} \backslash Z_{\Sigma_B} & \longrightarrow & P^B \\ \downarrow & & \downarrow \\ \mathbb{C}^E \backslash \{0\} & \longrightarrow & P^E \end{array}$$

The left vertical map is given on coordinates as

$$\alpha_i = x_i \prod_{\substack{I \in B \\ i \in I}} x_I.$$

For a homogeneous polynomial $\psi \in \mathbb{C}[\alpha_i \mid i \in E]$, we write $\pi_B^* \psi$ for the corresponding pullback.

Proposition 3.8.10. *Suppose $\psi \in \mathbb{C}[\alpha_i \mid i \in E]$ is homogeneous of degree d. Then the pullback $\pi_B^* \alpha$ has degree*

$$\deg(\pi_B^* \psi) = d \left[\sum_{i \in E} e_i + \sum_{i \in I \in B} e_I \right] \in \mathrm{Cl}(P^B)$$

for a fixed $i \in E$.

Proof. Note that ψ / α_i^d has degree zero, which means

$$\deg(\pi_B^*(\psi/\alpha_i^d)) = \deg(\pi_B^*(\psi)) - \deg(\pi_B^*(\alpha_i^d)) = 0.$$

Thus we can reduce to the case $\psi = \alpha_i^d$, in which case the assertion is clear from the above coordinate change. $\square$

3.9 Generalized permutahedra

The toric varieties we will consider later on turn out to be associated to very special polytopes, called *generalized permutahedra* in ([Pos09],[AA17]).

Consider first the building set $G_{max} = 2^E \backslash \{E\}$. Its fan $\Sigma_{G_{max}}$ is spanned by cones $\sigma = \mathrm{pos}(e_{I_1}, \ldots, e_{I_{n-1}})$ such that

$$I_0 = \emptyset \subsetneq I_1 \subsetneq \ldots \subsetneq I_{n-1} \subsetneq I_n = E$$

is a complete flag of subsets. On the other hand, let π_E be the convex hull of all points

$$a_\sigma = \sum_{k=1}^{n} k a_{\sigma(k)},$$

where σ runs over the bijections $\{1, \ldots, n\} \cong E$. The polytope π_E is the (regular) permutahedron of the finite set E. The following proposition is then well-known, see e.g. [Pos09].

Proposition 3.9.1. *The normal fan Σ_{π_E} of π_E coincides with $\Sigma_{G_{max}}$.*

The facet structure of π_E is very well understood. It is advocated in [AA17] to exploit this fact by expressing many questions in algebraic combinatorics in terms of deformations of π_E:

Definition 3.9.2. ([AA17]) A lattice polytope $Q \subseteq \mathbb{R}^E$ contained in an affine hyperplane

$$Q \subseteq \{m \in \mathbb{R}^E \mid \langle m, e^E \rangle = d_Q\}$$

is a *generalized permutahedron* if its normal fan is a coarsening of the fan Σ_{π_E}.

It will be convenient to have alternative characterizations of generalized permutahedra. Suppose $z : 2^E \to \mathbb{Z} \cup \{\infty\}$ is a set function with $z(\emptyset) = 0$. To z we associate the base polyhedron

$$Q(z) = \{m \in \mathbb{R}^E \mid \langle m, e^E \rangle = z(E),\ \langle m, e^I \rangle \geq z(I) \text{ for } I \subsetneq E\}.$$

We will call z *supermodular*, if

$$z(I) + z(J) \leq z(I \cap J) + z(I \cup J),$$

for all $I, J \in 2^E$.

Remark 3.9.3. It is more common in the literature to consider *submodular* functions $\tilde{z} : 2^E \to \mathbb{R} \cup \{\infty\}$, which satisfy the opposite inequality:

$$\tilde{z}(I) + \tilde{z}(J) \geq \tilde{z}(I \cap J) + \tilde{z}(I \cup J)$$

It is easy to show that $\tilde{z}$ is submodular if and only if its dual $\tilde{z}^{\#}$, defined as $\tilde{z}^{\#}(I) = \tilde{z}(E) - \tilde{z}(E \backslash I)$, is supermodular. The translation between the two convention is usually straightforward.

Proposition 3.9.4. *Let $Q \subseteq \mathbb{R}^E$ be a lattice polytope. Then the following are equivalent:*

1. *Q is a generalized permutahedron.*
2. *Every edge of Q is parallel to an edge of the form $e_i - e_j$ for $i, j \in E$.*
3. *There is a supermodular function $z : 2^E \to \mathbb{R}$, such that $Q = Q(z)$.*

Proof. See [AA17, Thm. 12.3] and references therein. □

Example 3.9.5. Let M be a matroid on the set E and $B(M) \subseteq 2^E$ its set of bases. We refer to [Oxl06] for the theory of matroids. The matroid polytope of M is

$$Q_M = \mathrm{Conv}(e^I \mid I \in B(M)).$$

It is proven in [GGMS87] that every edge of Q_M is of the form $e^i - e^j$, hence Q_M is a generalized permutahedron. The corresponding supermodular function is given by

$$z(I) = r_M(E) - r_M(E \backslash I) = r^{\#}(I),$$

where r_M is the rank function of the matroid.

Example 3.9.6. Let $\mathcal{G} \subseteq 2^E \backslash \{E\}$ be a building set and $\tilde{\mathcal{G}} = \mathcal{G} \cup \{E\}$. For $I \in \tilde{\mathcal{G}}$, let $\Delta_I = \mathrm{Conv}(e^i \mid i \in I)$ be the simplex on I and let $Q_{\mathcal{G}}$ be the Minkowski sum

$$Q_{\mathcal{G}} = \sum_{I \in \tilde{\mathcal{G}}} \Delta_I.$$

It is shown in [Pos09], that its normal fan is $\Sigma_{\mathcal{G}}$ and that $Q_{\mathcal{G}}$ is the base polyhedron of the supermodular function

$$z_{\mathcal{G}}(J) = |\{I \in \tilde{\mathcal{G}} \mid I \subseteq J\}|.$$

Hence $Q_{\mathcal{G}}$ is a generalized permutahedron and $\Sigma_{\mathcal{G}}$ is a coarsening of Σ_{π_E}.

Suppose $\mathcal{G}_1 \subseteq \mathcal{G}_2$ are two building sets in $2^E \backslash \{E\}$. It follows from the above description and Prop. 3.5.5 that $\Sigma_{\mathcal{G}_2}$ is a refinement of $\Sigma_{\mathcal{G}_1}$.

For $I \subsetneq E$ define the restriction $z|_I$ and contraction $z/_I$ by

$$\begin{aligned} z|_I(J) &= z(J), \quad J \subseteq I, \\ z/_I(J) &= z(J \cup I) - z(I), \quad J \subseteq E \backslash I. \end{aligned}$$

It is easy to check that if z is supermodular, then so are its restrictions and contraction. The face $F_{e^I} Q(z)$ can then be described as follows.

Proposition 3.9.7 ([Fuj91, Lemma 3.1])**.** *Let $Q(z)$ be the generalized permutahedron defined by the supermodular function $z : 2^E \to \mathbb{R}$. The natural isomorphism $\mathbb{R}^I \oplus \mathbb{R}^{I^c} \cong \mathbb{R}^E$ induces a bijection*

$$Q(z|_I) \times Q(z/_I) \cong F_{e^I} Q(z).$$

Example 3.9.8. If $z = r^{\#}$ is the dual of the rank function of a matroid M on E, then $z|_I$ and $z/_I$ correspond to the contraction $M/_{I^c}$ and restriction $M|_I$.

Let

$$\mathcal{I} : I_0 = \emptyset \subsetneq I_1 \subsetneq \ldots \subsetneq I_{n-1} \subsetneq I_n = E$$

be a maximal flag of 2^E. The corresponding cone $\sigma_{\mathcal{I}} = \mathrm{pos}(e^I \mid I \in \mathcal{I})$ is a maximal cone of Σ_{π_E}. Since Σ_{π_E} is a refinement of $\Sigma_{Q(z)}$, any vector $w \in \mathrm{Int}(\sigma)$ defines a unique vertex $m_{\mathcal{I}} = F_w Q(z)$ by Prop. 3.5.4.

Proposition 3.9.9 ([Fuj91, Corollary 3.17])**.** *The coordinates of the vertex $m_{\mathcal{I}}$ are given by*

$$(m_{\mathcal{I}})_k = z(I_k) - z(I_{k-1}).$$

Let us call a generalized permutahedron $Q(z)$ *irreducible*, if there is no nontrivial decomposition $E = I \sqcup J$, such that $z = z|_I + z|_J$.

Proposition 3.9.10 ([Fuj91, Thm. 3.38])**.** *For each generalized permutahedron $Q(z)$, there is a unique decomposition $E = \coprod_{k=1}^r I_k$ such that the $Q(z|_{I_k})$ are irreducible and*

$$z = \sum_{k=1}^r z|_{I_k}.$$

The polytope $Q(z)$ is irreducible if and only if it has maximal dimension $|E| - 1$.

Corollary 3.9.11. *Suppose $Q(z)$ is irreducible. A subset $I \subsetneq E$ defines a facet $F_{e^I} Q(z)$ of $Q(z)$ if and only if $Q(z|_I)$ and $Q(z/_I)$ are both irreducible.*

Let us use the preceding results to construct a smooth refinement of the normal fan of $Q(z)$. Consider the subset system

$$\tilde{\mathcal{G}}_z = \{I \subseteq E \mid Q(z|_I) \text{ is irreducible } \} \subseteq 2^E.$$

Proposition 3.9.12. *$\tilde{\mathcal{G}}_z$ is a building set in 2^E. If $Q(z)$ is irreducible, then the fan $\Sigma_{\mathcal{G}_z}$ associated to the reduced building set $\mathcal{G}_z = \mathcal{G} \backslash \{E\}$ is a smooth refinement of $\Sigma_{Q(z)}$.*

Proof. The building set property is immediate from the unique decomposition of Prop. 3.9.10. To prove that $\Sigma_{\mathcal{G}_z}$ is a refinement of $\Sigma_{Q(z)}$, we must prove that for each maximal nested set $\mathcal{I} \subseteq \mathcal{G}_z$, there is $m \in Q(z)$ such that $\langle m, e^I \rangle = z(I)$ for all $I \in \mathcal{I}$. By Example 3.8.5 we can find a maximal chain

$$J_0 = \emptyset \subsetneq J_1 \subsetneq \ldots \subsetneq J_{n-1} \subsetneq J_n = E$$

such that $\mathcal{I} = \bigcup \mathcal{I}_k$, where $\mathcal{I}_k = \max \mathcal{G}_z^{\subsetneq J_k}$. Let m be the vertex of $Q(z)$, defined by $m_k = z(J_k) - z(J_{k-1})$. Since $\mathcal{I}_k$ is the decomposition of J_k into irreducible components, we have

$$\sum_{I \in \mathcal{I}_k} z(I) = z(J_k) = \langle m, e^{J_k} \rangle = \sum_{I \in \mathcal{I}_k} \langle m, e^I \rangle.$$

Since $m \in Q(z)$, this equality is only possible if $\langle m, e^I \rangle = z(I)$ for all $I \in \mathcal{I}_k$. $\square$

4 D-modules

In this chapter, we collect some results from the theory of $\mathcal{D}$-modules on analytic manifolds, which will be needed for our discussion of the moderate cohomological functor. We refer to [HTT08],[Kas03] and [Bjö93] for detailed expositions.

4.1 Basic notions

Let X be a complex analytic manifold. The sheaf $\mathcal{D}_X$ of holomorphic differential operators is defined as a subsheaf of $\mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X)$ as follows: Let

$$F_0\mathcal{D}_X = \mathcal{O}_X \subseteq \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X),$$

where the action is given by multiplication, and define $F_k\mathcal{D}_X$ for $k > 0$ as

$$F_k\mathcal{D}_X = \{P \in \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X) \mid [P, f] \in F_{k-1}\mathcal{D}_X \text{ for all } f \in \mathcal{O}_X\}.$$

We then have $\mathcal{D}_X = \bigcup_{k\geq 0} F_k\mathcal{D}_X$. This gives $\mathcal{D}_X$ a natural structure as a filtered sheaf of rings, where the multiplication $F_k\mathcal{D}_X \otimes F_l\mathcal{D}_X \to F_{k+1}\mathcal{D}_X$ is given by composition of differential operators.

Remark 4.1.1. To ease notation, we will often write $P \in \mathcal{D}_X$ for a differential operator $P \in \mathcal{D}_X(U)$ defined over some open subset $U \subseteq X$.

A $\mathcal{D}_X$-*module* is an $\mathcal{O}_X$-module $M \in \mathrm{Mod}(\mathcal{O}_X)$ together with a module morphism

$$\mathcal{D}_X \otimes_{\mathbb{O}_X} M \to M,$$

satisfying the usual associativity conditions. We denote the category of $\mathcal{D}_X$-modules by $\mathrm{Mod}(\mathcal{D}_X)$.

Example 4.1.2. The sheaves $C^\infty_X, \mathcal{A}_X, \mathcal{O}_X$ and $\mathcal{B}_X$ of smooth, analytic, holomorphic and hyperfunctions all have a natural $\mathcal{D}_X$-module structure, given by differentiating sections.

Example 4.1.3. Let (M, ∇) be a locally free $\mathcal{O}_X$-module with an integrable connection ∇. Then M is a $\mathcal{D}_X$-module such that $\theta \cdot m = \nabla_\theta m$ for $\theta \in \Theta_X$. A $\mathcal{D}_X$-module is of this form if and only if it is coherent as an $\mathcal{O}_X$-module ([HTT08, Thm. 1.4.10.]).

Example 4.1.4. If $P_1, \ldots, P_m$ are analytic differential operators then the corresponding system of differential equations is encoded by the $\mathcal{D}_X$-module $M := \mathcal{D}_X/\mathcal{J}_P$ where

$$\mathcal{J}_P = \mathcal{D}_X\langle P_1, \ldots, P_m\rangle$$

is the left ideal generated by the P_i. The solutions to

$$P_i u = 0, \text{ for } i = 1, \dots, m$$

in the $\mathcal{D}_X$-module $\mathcal{L}$ are given by $\mathcal{S}ol_P(\mathcal{L}) = \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{L})$.

A $\mathcal{D}_X$-module M is called *coherent* if it is locally finitely generated and any submodule N of $M|_U$ for an open $U \subseteq X$ is locally finitely presented. The algebraic approach to partial differential equations essentially works because of the following

Theorem 4.1.5 ([Kas03, Prop. A.34])**.** *$\mathcal{D}_X$ is coherent over itself.*

The above theorem implies that a $\mathcal{D}_X$-module M is coherent if it is locally finitely presented.

Remark 4.1.6. We can always find holomorphic coordinates (U, x) inducing an isomorphism of $U \subseteq X$ with an open neighbourhood $V \subseteq \mathbb{C}^n$. The associated canonical vector fields ∂_i commute and every differential operator $P \in \mathcal{D}_X(U)$ of order k can be written in multi-index notation as

$$P = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$$

for $a_\alpha \in \mathcal{O}_X(U)$ and $\partial^\alpha = \prod_i \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$.

The ring $\mathcal{D}_X$ is not commutative, so we must distinguish between left and right $\mathcal{D}_X$-modules. Note that right $\mathcal{D}_X$-modules are the same thing as left modules of $\mathcal{D}_X^{op}$, the opposite ring of $\mathcal{D}_X$.

The rings $\mathcal{D}_X$ and $\mathcal{D}_X^{op}$ are generated by $\mathcal{O}_X \oplus \Theta_X$, where Θ_X is the $\mathcal{O}_X$-module of holomorphic vector fields on X. A (left or right) $\mathcal{D}_X$-module is then characterized by its $\mathcal{O}_X$-module structure and the action of Θ_X.

Proposition 4.1.7. *An $\mathcal{O}_X$-module M is a left $\mathcal{D}_X$-module if and only if there is an action*

$$\Theta_X \otimes M \to M,$$

satisfying the following conditions for all $f \in \mathcal{O}_X, s \in M$ and $\theta, \tilde{\theta} \in \Theta_X$:

1. $(f\theta) \cdot s = f(\theta \cdot s)$
2. $\theta \cdot (fs) = \theta(f)s + f(\theta \cdot s)$
3. $[\theta, \tilde{\theta}] \cdot s = \theta \cdot (\tilde{\theta} \cdot s) - \tilde{\theta} \cdot (\theta \cdot s)$

Similarly, a right $\mathcal{D}_X$-module N is given by an action

$$N \otimes \Theta_X \to N,$$

satisfying for all $f \in \mathcal{O}_X, s \in N$ and $\theta, \tilde{\theta} \in \Theta_X$:

1. $s \cdot (f\theta) = (fs) \cdot \theta$

2. $(fs)\cdot\theta = \theta(f)s + f(s\cdot\theta)$

3. $s\cdot[\theta,\tilde{\theta}] = (s\cdot\theta)\cdot\tilde{\theta} - (s\cdot\tilde{\theta})\cdot\theta$

We can then use the Leibniz rule to endow tensor products and morphism spaces with $\mathcal{D}_X$-module structures:

Proposition 4.1.8. *Let M_1, M_2 be left $\mathcal{D}_X$-modules and N_1, N_2 right $\mathcal{D}_X$-modules.*

1. *The sheaves $M_1 \otimes_{\mathcal{O}_X} M_2, \mathcal{H}om_{\mathcal{O}_X}(M_1, M_2)$ and $\mathcal{H}om_{\mathcal{O}_X}(N_1, N_2)$ are naturally left $\mathcal{D}_X$-modules, where the action of Θ_X is given by*

$$\begin{aligned}\theta\cdot(s_1\otimes s_2) &= \theta s_1\otimes s_2 + s_1\otimes\theta s_2\\ (\theta\varphi)(s) &= -\varphi(\theta s) + \theta(\varphi(s))\\ (\theta\varphi)(s) &= \varphi(s\theta) - (\varphi(s))\theta\end{aligned}$$

2. *The sheaves $N_1 \otimes_{\mathcal{O}_X} M_1$ and $\mathcal{H}om_{\mathcal{O}_X}(M_1, N_1)$ are right $\mathcal{D}_X$-modules, where the action of Θ_X is given by*

$$\begin{aligned}(s_1\otimes s_2)\cdot\theta &= s_1\theta\otimes s_2 - s_1\otimes s_2\theta\\ (\varphi\theta)(s) &= \varphi(s\theta) + (\varphi(s))\theta\end{aligned}$$

Proof. Checking the compatibility relations is straightforward, albeit a bit tedious. □

Remark 4.1.9. The $\mathcal{O}_X$-module adjunction

$$\mathcal{H}om_{\mathcal{O}_X}(M_1\otimes_{\mathcal{O}_X} M_2, M_3) \cong \mathcal{H}om_{\mathcal{O}_X}(M_1, \mathcal{H}om_{\mathcal{O}_X}(M_2, M_3))$$

is an isomorphism of $\mathcal{D}_X$-modules if M_1, M_2, M_3 are (say) left $\mathcal{D}_X$-modules.

Proposition 4.1.10 ([Bjö93, Prop. 2.1.3 and Lemma 2.2.5])**.** *For left $\mathcal{D}_X$-modules M_1, M_2, M_3 and a right $\mathcal{D}_X$-module N, there are isomorphisms of sheaves*

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X}(M_1, \mathcal{H}om_{\mathcal{O}_X}(M_2, M_3)) &\cong \mathcal{H}om_{\mathcal{D}_X}(M_1, \otimes_{\mathcal{O}_X} M_2, M_3)\\ N\otimes_{\mathcal{D}_X}(M_1\otimes_{\mathcal{O}_X} M_2) &\cong (N\otimes_{\mathcal{D}_X} M_1)\otimes_{\mathcal{O}_X} M_2.\end{aligned}$$

We will later need a way to switch between left and right modules. Let $\omega_X = \Omega_X^{\dim X}$ be the canonical sheaf of X, i.e. the sheaf of holomorphic differential forms of highest degree.

Proposition 4.1.11. *The canonical sheaf ω_X has a $\mathcal{D}_X^{op}$ structure given by*

$$\eta\cdot\theta = -\mathcal{L}_\theta\eta, \quad \theta\in\Theta_X, \eta\in\omega_X$$

Proof. Follows from the standard properties of the Lie derivative. □

Proposition 4.1.12 ([Kas03, Cor 1.11].)**.** *There is an exact equivalence of categories*

$$-\otimes_{\mathcal{O}_X}\omega_X : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X^{op}).$$

Its quasi-inverse is given by $-\otimes_{\mathcal{O}_X}\omega_X^{-1}$. *These functors are called the* side-changing operations.

Remark 4.1.13. Choosing local coordinates (x^i) on an open $U \subseteq X$ defines a nowhere vanishing section $\mathrm{d}x \in \omega_X(U)$ and thus an isomorphism $\omega_X|_U \cong \mathcal{O}_U$. Then the right action on $M \otimes_{\mathcal{O}_X} \omega_X$ is given by

$$(m \otimes \mathrm{d}x) \cdot P = P^t m \otimes \mathrm{d}x,$$

where P^t is the formal adjoint of P (with respect to the volume form $\mathrm{d}x$).

Similarly, let N be a right $\mathcal{D}_X$ module and let $\mathrm{d}x^{-1}$ be the section of ω_X^{-1} dual to $\mathrm{d}x$. Then the left action on $N \otimes_{\mathcal{O}_X} \omega_X^{-1} = \mathcal{H}om(\omega_X, N)$ is given by

$$P(n \otimes \mathrm{d}x^{-1}) = nP^t \otimes \mathrm{d}x^{-1}.$$

Remark 4.1.14. We will usually only state result for left modules. The corresponding version for right modules will then follow by the above equivalence.

4.2 Inverse and direct images

Let $f : X \to Y$ be a morphism. The sheaf

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$$

has a natural $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule structure. The right $f^{-1}\mathcal{D}_Y$ structure is the obvious one. The left $\mathcal{D}_X$ structure is defined by the pushforward map $f_* : \Theta_X \to f^{-1}\Theta_Y$: For $\theta \in \Theta_X$ and $g \otimes P \in \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ the action is given by

$$\theta \cdot (g \otimes P) = (\theta \cdot g) \otimes P + g \otimes f_*\theta P.$$

Example 4.2.1. Let $f : \mathbb{C}^k \cong \{0\}^{n-k} \times \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ be the local model for an immersion. Then

$$\mathcal{D}_{\mathbb{C}^k\to\mathbb{C}^n} = \mathcal{D}_{\mathbb{C}^k}[\partial_i \mid 1 \le i \le n-k]$$

Example 4.2.2. Let $f : \mathbb{C}^m \to \mathbb{C}^n$ for $m > n$ be the local model for a submersion. Then

$$\mathcal{D}_{\mathbb{C}^m\to\mathbb{C}^n} = \mathcal{D}_{\mathbb{C}^n}/\mathcal{D}_{\mathbb{C}^m}\langle \partial_i \mid n < i \le m\rangle$$

For a $\mathcal{D}_Y$-module M the pullback under f is defined by

$$f^*(M) = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}M.$$

This gives a right exact functor

$$f^* : \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_X)$$

which commutes with the forgetful functor $\mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{O}_X)$.

Example 4.2.3. For $M \in \mathrm{Mod}(\mathcal{D}_Y)$ we have $f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$ as an $\mathcal{O}_X$-module. For a local coordinate system (y_i, ∂_i) on Y, we can describe the action of $\theta \in \Theta_X$ on f^*M as

$$\theta \cdot (\phi \otimes s) = \theta(\phi) \otimes s + \phi \sum_{i=1}^{n} \theta(y_i \circ f) \otimes \partial_i s.$$

For a *right* $\mathcal{D}_X$-module N, we can use the above transfer bimodule to define the direct image

$$f_*(N \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \in \mathrm{Mod}(\mathcal{D}_Y^{op}).$$

For direct images of left modules, we need a side switched version of $\mathcal{D}_{X\to Y}$. Define the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\mathcal{D}_{Y\leftarrow X}$ by

$$\begin{aligned}\mathcal{D}_{Y\leftarrow X} &:= \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1} \\ &= \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1})\end{aligned}$$

The direct image of a $\mathcal{D}_X$-module M is then

$$f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} M).$$

This is not very well behaved in general since it involves a mixture of left and right exact functors. A satisfactory theory of these functors requires working in the derived category.

Remark 4.2.4. The two model cases of embeddings and smooth maps are often enough to understand the formal properties of the direct and inverse image functors, since we can decompose every map $f : X \to Y$ into the the graph embedding $\Gamma_f : X \to X \times Y$ and the projection $\pi : X \times Y \to Y$.

4.3 Derived category of D-modules

Let $\mathrm{Mod}(\mathcal{D}_X)$ be the category of $\mathcal{D}_X$-modules on X and $\mathrm{Mod}_c(\mathcal{D}_X)$ its subcategory of coherent modules. Denote by $D^b(\mathcal{D}_X)$ and $D^b_c(\mathcal{D}_X)$ the corresponding bounded derived categories.

Every (left or right) $\mathcal{D}_X$-module has a flat resolution of length at most $\dim_{\mathbb{C}} X$ ([Bjö93, Prop. 2.21]). Hence we can construct derived tensor products

$$- \otimes^L_{\mathcal{D}_X} - : D^b(\mathcal{D}_X) \times D^b(\mathcal{D}_X^{op}) \to D^b(\mathbb{C}_X).$$

Since $\mathcal{D}_X$ has finite cohomological dimension, we also get derived morphism spaces

$$R\,\mathrm{Hom}_{\mathcal{D}_X}(-,-) : D^b(\mathcal{D}_X)^{op} \times D^b(\mathcal{D}_X) \to D^b(\mathbb{C}_X),$$

as well as an analogous version for right modules. We can us the same (flat or injective) resolutions to construct the derived functors $- \otimes^L_{\mathcal{O}_X} -$ and $R\mathcal{H}om_{\mathcal{O}_X}(-,-)$. Derived versions of the adjunctions in Remark 4.1.9 and Prop. 4.1.10 can then be constructed using appropriate resolutions.

Proposition 4.3.1. *There are locally free resolutions of $\mathcal{D}_X$- resp. $\mathcal{D}_X^{op}$-modules:*

$$\mathcal{O}_X \cong \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{*} \Theta_X$$
$$\omega_X \cong \Omega_X^* \otimes_{\mathcal{O}_X} \mathcal{D}_X[\dim X]$$

Proof sketch. In local coordinates, the differential are described as

$$\begin{aligned} d : \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k} \Theta_X &\to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k+1} \Theta_X, \\ d(P \otimes \theta_1 \wedge \ldots \wedge \theta_k) &= \sum_i (-1)^{i-1} (P\theta_i) \otimes \theta_1 \wedge \ldots \wedge \widehat{\theta}_i \wedge \ldots \wedge \theta_k \\ &+ \sum_{i<j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \theta_1 \wedge \ldots \wedge \widehat{\theta}_i \wedge \ldots \wedge \widehat{\theta}_j \wedge \ldots \wedge \ldots \wedge \theta_k \end{aligned}$$

and

$$\begin{aligned} d : \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{D}_X &\to \Omega_X^{k+1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ d(\eta \otimes P) &= d\eta \otimes P + \sum_i \mathrm{d}z_i \wedge \eta \otimes \partial_i P. \end{aligned}$$

The natural maps

$$\mathcal{D}_X \to \mathcal{O}_X = \mathcal{D}_X / \mathcal{D}_X \cdot \Theta_x$$

and

$$\omega_X \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

furnish the required morphism of complex. Note that the second is deduced from the first by side-changing, so it is enough to prove that the first complex is quasi-isomorphic to $\mathcal{O}_X$. This is shown in [HTT08, Lemma 1.5.27] and in [Bjö93, Section 1.5.4]. □

Example 4.3.2. Let $(M, \nabla) \in \mathrm{Mod}(\mathcal{D}_X)$ be an integral connection. Then

$$\omega_X \otimes^L_{\mathcal{D}_X} M[\dim X] \cong (\Omega_X^* \otimes_{\mathcal{O}_X} M, d_\nabla)$$

is the de Rahm complex associated to (M, ∇). Here d_∇ is the extension of ∇ as a graded derivation:

$$d_\nabla(\eta \otimes m) = d\eta \otimes m + (-1)^{\deg(\eta)} \eta \wedge \nabla m.$$

Its cohomology is concentrated in degree 0 and

$$H^0(\omega_X \otimes^L_{\mathcal{D}_X} M[\dim X]) = \ker \nabla$$

is the locally constant sheaf of horizontal sections.

4.4 Derived pullback

Now let $f : X \to Y$ be a morphism of varieties or complex manifolds. We can define a derived pullback as

$$\begin{aligned} \underline{f}^* : D^b(\mathcal{D}_Y) &\to D^b(\mathcal{D}_Y) \\ \underline{f}^* M &\mapsto \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}M. \end{aligned}$$

The derived pullback does not preserve coherency in general. This can already be seen in Example 4.2.1.

Example 4.4.1. Let $f : X \to Y$ be a smooth morphism. Then $\mathcal{O}_X$ is a flat $f^{-1}\mathcal{O}_Y$-module. It follows that $H^j\underline{f}^*$ vanishes for $j \neq 0$ and f^* is an exact functor, since this is true for the underlying $\mathcal{O}_X$-modules. In this case, f also preserves coherency.

Example 4.4.2. Let $f = i : X \to Y$ be a closed embedding of complex codimension $d_{Y/X}$. The associated Koszul-complex $K^* \cong i^{-1}\mathcal{O}_Y$ gives a resolution of $\mathcal{O}_X$ by locally free $i^{-1}\mathcal{O}_Y$-modules. Tensoring with $i^{-1}\mathcal{D}_Y$ gives the resolution

$$K^* \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y \cong \mathcal{D}_{X\to Y},$$

where the left hand side is a complex of locally free $i^{-1}\mathcal{D}_Y$-modules.

For $M \in \mathrm{Mod}(\mathcal{D}_Y)$ we then get

$$\underline{i}^*(M) = K^* \otimes_{i^{-1}\mathcal{O}_Y} M.$$

It follows that $H^j\underline{i}^*(M) = 0$ unless $-d_{Y/X} \leq j \leq 0$.

Example 4.4.3. Let $p_i : X \times X \to X, i = 1, 2$ be the natural projections and $\Delta : X \to X \times X$ the diagonal embedding. We have the natural isomorphism in $D^b(\mathcal{D}_X)$

$$M \otimes^L_{\mathcal{O}_X} N \cong \underline{\Delta}^*(M \boxtimes N),$$

where

$$M \boxtimes N := p^{-1}M \otimes_{p_1^{-1}\mathcal{O}_X} \mathcal{O}_{X\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}N$$

is the external tensor product of $\mathcal{D}_X$-modules.

4.5 Derived direct images

Let again $f : X \to Y$ be a morphism of complex manifolds. The derived version of the direct image functor is

$$\begin{aligned} \underline{f}_* : D^b(\mathcal{D}_X) &\to D^b(\mathcal{D}_Y) \\ \underline{f}_* M &= Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} M). \end{aligned}$$

Example 4.5.1. For an open embedding $j : U \to Y$ we have $\mathcal{D}_{Y\to U} = \mathcal{D}_U$ and hence $\underline{j}_* M = j_* M$.

Example 4.5.2. Let $i : X \to Y$ be a closed embedding and $M \in \mathrm{Mod}(\mathcal{D}_X)$. For a local coordinate system (y_i, ∂_i) such that $X = \{y_{r+1} = \ldots = y_n = 0\}$, we have

$$\underline{i}_* M = \mathbb{C}[\partial_{r+1}, \ldots, \partial_n] \otimes_{\mathbb{C}} i_* M.$$

See [HTT08, Prop. 1.5.24].

Example 4.5.3. Let $p : X = Y \times Z \to Y$ be the projection. We have

$$\mathcal{D}_{Y \leftarrow Y \times Z} = \mathcal{D}_Y \boxtimes \omega_Z$$

Using the resolution $\omega_Z \cong \Omega_Z^* \otimes_{\mathcal{O}_Z} \mathcal{D}_Z[\dim Z]$ from Prop. 4.3.1 gives

$$\begin{aligned} \mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} M &= \Omega^*_{X/Y} \otimes_{\mathcal{O}_X} M[\dim X/Y] \\ &=: DR_{X/Y}(M), \end{aligned}$$

where

$$\Omega^k_{X/Y} = \Omega^k_X / p^* \Omega^k_Y \cong \mathcal{O}_X \otimes_{p_Z^{-1}\mathcal{O}_Z} \Omega^k_Z$$

are the relative differential forms. The pushforward is then

$$\underline{p}_* M = Rp_*(DR_{X/Y} M).$$

4.6 Characteristic varieties

We have seen above, that the ring $\mathcal{D}_X$ has a natural exhaustive filtration $F_k\mathcal{D}_X$ such that $F_0\mathcal{D}_X = \mathcal{O}_X$ and $F_1\mathcal{D}_X = \Theta_X \bigoplus \mathcal{O}_X$.

The associated graded is

$$gr_F \mathcal{D}_X = \bigoplus_{k=0}^{\infty} \pi_* \mathcal{O}_{T^*X}(k),$$

where $\pi_*\mathcal{O}_{T^*X}(k) \subseteq \pi_*\mathcal{O}_{T^*X}$ denotes the subsheaf of holomorphic functions which are polynomial of degree k in the fiber variables.

A good filtration of a coherent $\mathcal{D}_X$-module M consists of an increasing, locally bounded below sequence $F_k M \subseteq M$ of $\mathcal{O}_X$-submodules such that $gr^F M$ is coherent over $gr^F \mathcal{D}_X$. We can then define the coherent $\mathcal{O}_{T^*X}$-module

$$\widetilde{gr^F M} := \pi^{-1} gr^F M \otimes_{\mathcal{O}_{\pi^{-1}X}} \mathcal{O}_{T^*X}$$

Proposition 4.6.1 ([Bjö93, Section 1.6.18])**.** *Every coherent $\mathcal{D}_X$-module locally has a good filtration. The support of $\widetilde{gr^F M}$ does not depend on the chosen filtration. It is a closed $\mathbb{C}^*$-conic, and involutive subvariety of T^*X of dimension at least* $\dim X$.

Definition 4.6.2. The support of $\widetilde{gr^F M}$ is called the *characteristic variety* of M and denoted $Ch(M)$.

The characteristic variety of a $\mathcal{D}_X$-module is naturally related to the microsupport of sheaves defined in Section 2.6.

Theorem 4.6.3 ([KS94, Thm. 11.3.3])**.** *Let M be a coherent $\mathcal{D}_X$-module and*

$$Sol_X(M) = R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)$$

its holomorphic solution complex. Then

$$SS(Sol_X(M)) = Ch(M).$$

5 Distributions and hyperfunctions

In this chapter, we apply the results of chapters 2 and 4 to the study of distributions. First we review the moderate cohomology functor of Kashiwara and Schapira ([KS96]), as well as Sato's theory of hyperfunctions. and apply it to the study of boundary value representations of distributions. The last sections provide numerous constructions of distributions, which will be needed for our construction of the analytically regularized amplitudes.

5.1 Moderate and formal cohomology

Let X be a real analytic manifold and denote by $\mathcal{S}_X$ the partially ordered set of subanalytic, relatively compact open subsets. Let $\mathbb{R}-Cons(X)$ be the category of $\mathbb{R}$-constructible sheaves and $\mathbb{R}-Cons_c(X)$ be the subcategory of sheaves with compact support. There is a natural inclusion

$$\begin{aligned} i : \mathcal{S}_X &\to \mathbb{R}-Cons_c(X) \\ U &\mapsto \mathbb{C}_U. \end{aligned}$$

Let $\mathcal{A}$ be some abelian category and suppose we are given a map $\psi : \mathcal{S}_X \to \mathcal{A}$, such that

1. $\psi(\emptyset) = 0$
2. If $U, V \in \mathcal{S}_X$ then the sequence
$$0 \longrightarrow \psi(U \cap V) \longrightarrow \psi(U) \oplus \psi(V) \longrightarrow \psi(U \cup V) \longrightarrow 0$$
is exact.
3. If $U \subseteq V$ is an inclusions of open sets in $\mathcal{S}_X$, then $\psi(U) \hookrightarrow \psi(V)$ is a monomorphism.

Theorem 5.1.1 ([KS96, Thm. 1.1]). *Let $\psi : \mathcal{S}_X \to \mathcal{A}$ be a functor satisfying the above conditions.*

1. *There is an exact functor $\Psi : \mathbb{R}-Cons(X) \to \mathcal{A}$, unique up to isomorphism, such that $\Psi(\mathbb{C}_U) \cong \psi(U)$ for all $U \in \mathcal{S}_X$.*
2. *If $\tilde{\psi} : \mathcal{S}_X \to \mathcal{A}$ is also satisfying the above conditions, and $\theta : \psi \to \tilde{\psi}$ is a natural transformation, then there is a unique extension of θ to a natural transformation $\Theta : \Psi \to \tilde{\Psi}$.*

3. *If $\mathcal{A} \subseteq \mathrm{Mod}(\mathbb{C}_X)$ is a subcategory of the category of sheaves on X, then Ψ is local in the sense that $\Psi(F)|_U \cong \Psi(F_U)|_U$ for all $U \in \mathcal{S}_X$.*

By exactness of Ψ, we obtain a functor of triangulated categories $\Psi : D^b_{\mathbb{R}-c}(\mathbb{C}_X) \to D^b(\mathcal{A})$, which will be our main object of study. The following technical results is then often useful to reduce to the case $\Psi(\mathbb{C}_U)$ or $\Psi(\mathbb{C}_Z)$, where U (resp. Z) is subanalytic open (resp. closed) subset of X.

Proposition 5.1.2 ([KS96, Prop. 1.5]). *Let $\Psi_1, \Psi_2 : D^b_{\mathbb{R}-c}(\mathbb{C}_X) \to D^b(\mathcal{A})$ be morphisms of triangulated categories and $\Theta : \Psi_1 \to \Psi_2$ a natural transformation. Assume the following conditions:*

1. *If $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, then $\Theta(F)$ is an isomorphism if $\Theta(F_Z)$ is an isomorphism for any subanalytic, compact subset $Z \subseteq X$.*

2. *For any open (resp. closed) subset U (resp. Z), $\Theta(\mathbb{C}_U)$ (resp. $\Theta(\mathbb{C}_Z)$) is an isomorphism.*

Then $\Theta : \Psi_1 \to \Psi_2$ is an isomorphism of functors.

Let Db_X be the sheaf of distributions on X. For a subanalytic open subset $U \subseteq X$ and $Z = X \backslash U$ let

$$\mathcal{T}hom(\mathbb{C}_U, Db_X) := Db_X / \Gamma_Z Db_X.$$

This gives a functor

$$\mathcal{T}hom(-, Db_X) : \mathcal{S}_X^{op} \to \mathrm{Mod}(\mathcal{D}_X)$$

It was shown by Lojaciewicz, that this functor satisfy the Mayor-Vietoris property: For U, V two subanalytic open subsets, we have an exact sequence

$$0 \longrightarrow \mathcal{T}hom(\mathbb{C}_{U\cap V}, Db_X) \longrightarrow \mathcal{T}hom(\mathbb{C}_U, Db_X) \oplus \mathcal{T}hom(\mathbb{C}_V, Db_X)$$

$$\longrightarrow \mathcal{T}hom(\mathbb{C}_{U\cup V}, Db_X) \longrightarrow 0$$

The other two conditions in Theorem 5.1.1 are immediate from the constructions. We then obtain:

Theorem 5.1.3. *The above definition extends to an exact functor*

$$\mathcal{T}hom(-, Db_X) : \mathbb{R} - Cons(X)^{op} \to \mathrm{Mod}(\mathcal{D}_X),$$

where $\mathcal{D}_X$ denotes the sheaf of differential operators with real analytic *coefficients.*

Example 5.1.4. Let $Z \subseteq X$ be a subanalytic closed subset. Essentially by definition we have

$$\mathcal{T}hom(\mathbb{C}_Z, Db_X) = \Gamma_Z Db_X.$$

Example 5.1.5. Let $U \subseteq X$ be a subanalytic open subset. A section

$$u \in \Gamma(U, Db_X) \cong \Gamma(X, \mathcal{H}om(\mathbb{C}_U, Db_X))$$

lies in $\Gamma(X, \mathcal{T}hom(\mathbb{C}_U, Db_X))$ if and only if it extends to a global section $\tilde{u} \in \Gamma(X, Db_X)$. If $U \subseteq \mathbb{R}^n$ is relatively compact, then this is equivalent to u having moderate growth in the following sense: There exists $C > 0$ and $m, r \in \mathbb{N}$ such that

$$\langle u, \phi \rangle \leq C \sum_{|\alpha| \leq m} \sup_{x \in U} \left(d(x, \partial U)^{-r} |D^\alpha \phi(x)| \right)$$

for all $\phi \in \Gamma_c(U, C^\infty_{\mathbb{C}^n})$. See e.g. [Kas84, Lemma 3.3].

Example 5.1.6. Let $j : \mathbb{R}^n \to S^n$ be the one-point compactification of $\mathbb{R}^n$ and denote by $\infty \in S^n$ the point at infinity. Then

$$\Gamma(S^n, \mathcal{T}hom(j_! \mathbb{C}_{\mathbb{R}^n}, Db_{S^n})) = \Gamma(S^n, Db_{S^n} / \Gamma_{\{\infty\}} Db_X)$$

are the tempered distributions.

For a locally free $\mathcal{A}_X$-modules L and F a constructible sheaf we set

$$\mathcal{T}hom(F, Db_X \otimes_{\mathcal{A}_X} L) = \mathcal{T}hom(F, Db_X) \otimes_{\mathcal{A}_X} L$$

For $L = \mathcal{A}_X^{\dim_{\mathbb{R}} X} \otimes or_X$ the sheaf of analytic densities, we abbreviate

$$\mathcal{T}hom(F, Db_X^\vee) := \mathcal{T}hom(F, Db_X) \otimes_{\mathcal{A}_X} \mathcal{A}_X^{\dim_{\mathbb{R}} X} \otimes or_X$$

By exactness, we obtain a triangulated functor

$$\mathcal{T}hom(-, Db_X) : D^b_{\mathbb{R}-c}(\mathbb{C}_X) \to D^b(\mathcal{D}_X).$$

Remark 5.1.7. The above examples show that for $F = \mathbb{C}_S$ the sheaf corresponding to an open or closed subanalytic set, the functor $\mathcal{T}hom(F, Db_X)$ is an object of classical analysis. But for general $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, it is not clear how to think about $\mathcal{T}hom(F, Db_X)$. Using Prop. 5.1.2 it is often enough to consider the case $F = \mathbb{C}_U$.

Now let $f : Y \to X$ be a morphism of real analytic manifolds. Analogous to the complex case, we can define transfer bimodules by

$$\begin{aligned} \mathcal{D}_{Y \to X} &= \mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X \\ \mathcal{D}_{X \leftarrow Y} &= \mathcal{A}_Y^\vee \otimes_{\mathcal{A}_Y} \mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{A}_X} (f^{-1}\mathcal{A}_X^\vee)^{-1} \end{aligned}$$

For $M \in D^b(\mathcal{D}_X)$ we define the derived inverse image as

$$\underline{f}^{-1} M = \mathcal{D}_{Y \to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1} M.$$

Similarly, for a $N \in D^b(\mathcal{D}_Y)$ let

$$\underline{f}_* N = Rf_*(\mathcal{D}_{X \leftarrow Y} \otimes^L_{\mathcal{D}_Y} N)$$
$$\underline{f}_! N = Rf_!(\mathcal{D}_{X \leftarrow Y} \otimes^L_{\mathcal{D}_Y} N)$$

be the derived (proper) direct images.

Recall that there is a natural pushforward map of distributions

$$f_! Db_X^\vee \to Db_Y^\vee$$

given by

$$\langle f_! u, \phi \rangle = \langle u, f^* \phi \rangle,$$

for $\phi \in \Gamma_c(U, C_Y^\infty)$. This construction extends in our current setting as follows:

Proposition 5.1.8 ([KS96, Prop. 4.3]). *For $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ there is a natural morphism in $D^b(\mathcal{D}_X^{op})$:*

$$\underline{f}_! \mathcal{T}hom(f^{-1}F, Db_Y^\vee) \to \mathcal{T}hom(F, Db_X^\vee)$$

Proof Sketch. Let us first consider the case $F = \mathbb{C}_Z$, where Z is a subanalytic closed subset. An analog of Prop. 4.3.1 gives the Spencer resolution $Sp_*(\mathcal{D}_{Y \to X}) \cong \mathcal{D}_{Y \to X}$, where

$$Sp_*(\mathcal{D}_{Y \to X}) = \mathcal{D}_Y \otimes_{\mathcal{A}_Y} \bigwedge^{\bullet} \Theta_Y \otimes_{\mathcal{A}_Y} \mathcal{D}_{Y \to X}$$

We then have the isomorphism

$$\underline{f}_! \mathcal{T}hom(f^{-1}\mathbb{C}_Z, Db_Y^\vee) \cong f_! \mathcal{K}^*,$$

where

$$\mathcal{K}^m = \Gamma_{f^{-1}(Z)} Db_Y^\vee \otimes_{\mathcal{A}_Y} \bigwedge^{*} \Theta_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$$

and we have used that $\Gamma_{f^{-1}(Z)} Db_Y^\vee$ is c-soft. A small computation (see loc. cit) shows that the integration map

$$f_! \Gamma_{f^{-1}(Z)} Db_Y^\vee \to \Gamma_Z Db_X^\vee$$

gives a morphism of complexes

$$f_! \mathcal{K}^* \to \Gamma_Z Db_X^\vee.$$

Since the functor $F \mapsto f_!(\mathcal{T}hom(F, Db_Y^\vee) \otimes_{\mathcal{A}_Y} \bigwedge^* \Theta_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X)$ is exact for $F \in \mathbb{R} - Cons_X$, we can extend the construction to all $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. $\square$

Remark 5.1.9. The above proof constructs in particular an integration morphism

$$f_! \mathcal{T}hom(f^{-1}F, Db_Y^\vee) \to \mathcal{T}hom(F, Db_X^\vee)$$

which factors as

$$\begin{aligned} f_! \mathcal{T}hom(f^{-1}F, Db_Y^\vee) &\to f_! \left(\mathcal{T}hom(f^{-1}F, Db_Y^\vee) \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y \to X} \right) \\ &= \underline{f}_! \mathcal{T}hom(f^{-1}F, Db_Y^\vee) \\ &\to \mathcal{T}hom(F, Db_X^\vee). \end{aligned}$$

Proposition 5.1.10 ([KS96, Thm. 4.4]). *Let $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ such that f is proper on supp(G). Then the composition*

$$\underline{f}_! \mathcal{T}hom(G, Db_Y) \to \underline{f}_! \mathcal{T}hom(f^{-1}Rf_*G, Db^\vee_Y) \to \mathcal{T}hom(Rf_*G, Db^\vee_X)$$

is an isomorphism.

Example 5.1.11. Suppose $f : Y \hookrightarrow X$ is the inclusion of a submanifold of codimension d and $F = \mathbb{C}_Y$. Let $\mathcal{D}_{Y|X}$ be the subbundle of $f^{-1}\mathcal{D}_X$, generated by vector fields normal to Y. Then the above map takes the form

$$\begin{aligned} \underline{f}_! Db^\vee_Y = f_* Db^\vee_Y \otimes_{\mathcal{A}_Y} \mathcal{D}_{Y|X} &\to \Gamma_Y Db^\vee_X \\ u \otimes P &\mapsto f_! uP. \end{aligned}$$

Especially for both Y and X oriented and $P = 1$, we get

$$u \mapsto u \wedge \chi_Y,$$

where $\chi_Y \in Db^d_X$ is the current of the submanifold Y (cf. Prop. 5.5.10).

The morphism

$$\begin{aligned} \underline{f}_! \mathcal{T}hom(f^{-1}F, Db^\vee_Y) &\cong Rf_! \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_Y} \mathcal{T}hom(f^{-1}F, Db^\vee_Y) \\ &\to \mathcal{T}hom(F, Db^\vee_X) \end{aligned}$$

defines by side-changing and adjunctions the morphisms

$$\begin{aligned} \mathcal{D}_{X\leftarrow Y} \otimes^L_{\mathcal{D}_Y} \mathcal{T}hom(f^{-1}F, Db_Y) &\to f^! \mathcal{T}hom(F, Db_X) \\ \mathcal{T}hom(f^{-1}F, Db_Y) &\to R\mathcal{H}om_{f^{-1}\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, f^! \mathcal{T}hom(F, Db_X)) \end{aligned}$$

The above constructions are also compatible with external products.

Proposition 5.1.12. *Let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then there is a natural product morphism*

$$(\mathcal{T}hom(G, Db_X)) \boxtimes (\mathcal{T}hom(G, Db_X)) \to (\mathcal{T}hom(F \boxtimes G, Db_X))$$

Proof. We can reduce to the case $F = \mathbb{C}_U, G = \mathbb{C}_V$ for subanalytic open subsets U and V. In this case the construction is obvious. $\square$

Now let X be a complex manifold. We write $X_\mathbb{R}$ if we want to consider X as real a analytic manifold. The complex conjugate $\overline{X}$ is a complex manifold which has $X_\mathbb{R}$ as underlying real manifold and structure sheaf $\mathcal{O}_{\overline{X}}$, the sheaf of anti-holomorphic functions on X. The diagonal embedding

$$X_\mathbb{R} \hookrightarrow X \times \overline{X}$$

is a complexification of X.

We now write $\mathcal{D}_{X_\mathbb{R}}$ for the sheaf of real-analytic differential operators on $X_\mathbb{R}$, i.e. we have

$$\mathcal{D}_{X_\mathbb{R}} = (\mathcal{D}_X \boxtimes \mathcal{D}_{\overline{X}})|_{X_\mathbb{R}}$$

and the two subrings $\mathcal{D}_X, \mathcal{D}_{\overline{X}} \subseteq \mathcal{D}_{X_\mathbb{R}}$ commute.

Definition 5.1.13. Let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. The functors of moderate cohomology with respect to F is defined as

$$\mathcal{T}hom(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, Db_X)).$$

It is a natural object of $D^b(\mathcal{D}_X)$.

Recall from Prop. 4.3.1 the resolution $\mathcal{O}_{\overline{X}} \cong \mathcal{D}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \bigwedge^* \Theta_{\overline{X}}$. Applying $\mathcal{T}hom(-, Db_X)$ gives the isomorphism

$$\mathcal{T}hom(F, \mathcal{O}_X) \cong (\Omega^\bullet_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{T}hom(F, Db_X), \overline{\partial}),$$

Hence for F, a constructible sheaf, we can compute $\mathcal{T}hom(F, \mathcal{O}_X)$ as the Dolbeault complexes with values in $\mathcal{T}hom(F, Db_X)$.

Remark 5.1.14. For $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, there is a natural map

$$\mathcal{T}hom(F, Db_X) \to R\mathcal{H}om(F, Db_X)$$

by [KS96, Prop. 28]. Applying $R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, -)$ gives the natural map

$$\mathcal{T}hom(F, \mathcal{O}_X) \to R\mathcal{H}om(F, \mathcal{O}_X).$$

Let $f : Y \to X$ be a morphism of complex manifolds. By [KS96, Lemma 5.5], we have an isomorphism

$$\underline{f}_! R\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, N)[\dim_{\mathbb{C}} Y] \cong R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \underline{f}_{\mathbb{R}!} N)[\dim_{\mathbb{C}} X]$$

Applying this for $N = \mathcal{T}hom(f^{-1}F, Db_Y)$ together with a side-changed version of Prop. 5.1.8 gives the following.

Theorem 5.1.15 ([KS96, Thm 5.6])**.** *For $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ there is a natural morphism in $D^b(\mathcal{D}_X)$:*

$$\underline{f}_! \mathcal{T}hom(f^{-1}F, \omega_Y)[\dim_{\mathbb{C}} Y] \to \mathcal{T}hom(F, \omega_X)[\dim_{\mathbb{C}} X]$$

Remark 5.1.16. As in remark 5.1.9, this morphism factors as

$$f_! \mathcal{T}hom(f^{-1}F, \omega_Y)[\dim_{\mathbb{C}} Y] \to \underline{f}_! \mathcal{T}hom(f^{-1}F, \omega_Y)[\dim_{\mathbb{C}} Y] \to \mathcal{T}hom(F, \omega_X)[\dim_{\mathbb{C}} X].$$

We also have similar results for external products and pullbacks.

Theorem 5.1.17 ([KS96])**.** *Let X, Y be two complex manifolds, $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. There is a natural product map*

$$\mathcal{T}hom(F, \mathcal{O}_X) \boxtimes \mathcal{T}hom(G, \mathcal{O}_Y) \to \mathcal{T}hom(F \boxtimes G, \mathcal{O}_{X \times Y})$$

Proof. The map is obtained by applying $R\mathcal{H}om_{\mathcal{D}_{\overline{X}} \times \mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{X} \times \overline{Y}}, -)$ to the morphism in Prop. 5.1.12. □

Theorem 5.1.18 ([KS96, Prop. 5.9])**.** *For $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ there is a natural morphism*

$$\underline{f}^{-1}\mathcal{T}hom(F, \mathcal{O}_X) \to \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y).$$

Let DFN be the category of topological vector spaces of type DFN (duals of Frechet nuclear spaces) and let $D^b(DFN)$ be its derived category.

Theorem 5.1.19 ([KS96, Thm. 5.2 and Thm. 6.1])**.** *Let $F \in D_{\mathbb{R}-c}(\mathbb{C}_X)$. The space $R\Gamma_c(X, \mathcal{T}hom(F, \mathcal{O}_X))$ is naturally an object of $D^b(DFN)$, functorially with respect to F.*

Remark 5.1.20. The category DFN is not abelian, but only quasi-abelian. The structure of the derived category $D^b(DFN)$ is studied in detail in [Sch99].

5.2 Hyperfunctions

Let M be an m-dimensional real analytic manifold. A complexification of M is given by an analytic embedding

$$i : M \hookrightarrow X$$

into a complex manifold X, which is locally isomorphic to the embedding

$$\mathbb{R}^m \hookrightarrow \mathbb{C}^m.$$

The "germ" of such a complexification is unique in the following sense: If X' is another complexification, then there are neighbourhoods $W \subseteq X$ and $W' \subseteq X'$ of M, and a unique biholomorphism $W \cong W'$ extending the identity on M.

Denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, by $\mathcal{A}_M \cong \mathcal{O}_X|_M$ the sheaf of analytic functions on M, by $\pi : T^*_M X \to M$ the conormal bundle and by $\mathring{\pi} : \mathring{T}^*_M X \to M$ the restriction to $\mathring{T}^*_M X = T^*_M \backslash T^*_X X$, the set of nonvanishing covectors.

Definition 5.2.1.

1. The sheaf of *hyperfunctions* $\mathcal{B}_M \in D^b(\mathbb{C}_M)$ is defined as

$$\mathcal{B}_M = \mathcal{H}^m_M(\mathcal{O}_X) \otimes or_M.$$

2. The sheaf of *microfunctions* $\mathcal{C}_M \in D^b(\mathbb{C}_{T^*_M X})$ is

$$\mathcal{C}_M = \mathcal{H}^m(\mu_M(\mathcal{O}_X)) \otimes or_{M/X}.$$

Proposition 5.2.2 ([KS94, Prop. 11.5.2])**.** *Let M a real analytic manifold with complexification X.*

1. *The complexes $\mu_M(\mathcal{O}_X)$ and $R\Gamma_M(\mathcal{O}_X)$ are concentrated in degree m, i.e. we have natural isomorphisms*

$$\mathcal{B}_M \cong R\Gamma_M(\mathcal{O}_X) \otimes or_{M/X}[m] \cong R\mathcal{H}om(\omega_{M/X}, \mathcal{O}_X),$$
$$\mathcal{C}_M \cong \mu_M(\mathcal{O}_X) \otimes or_{M/X}[m] \cong \mu hom(\omega_{M/X}, \mathcal{O}_X).$$

2. *There is a natural isomorphism* $sp : \pi_* \mathcal{C}_M \to \mathcal{B}_M$.

3. *The sheaf* $\mathcal{B}_M$ *is flabby.*

4. *The sheaf* $\mathcal{C}_M|_{\overset{\circ}{T}{}^*_M X}$ *is conically flabby, i.e. its direct image on* $\overset{\circ}{T}{}^*_M X/\mathbb{R}^+$ *is flabby.*

5. *There is a long exact sequence*

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{B}_M \longrightarrow \mathring{\pi}_* \mathcal{C}_M \longrightarrow 0$$

Remark 5.2.3. Let $V \subseteq T^*_M X$ be a conic open set. Thm. 2.5.8 gives the isomorphism

$$\mathcal{C}_M(V) = \varinjlim_{U,Z} H^m_Z(U; \mathcal{O}_X \otimes or_{M/X})$$

where the colimit runs over open neighbourhoods $U \subseteq X$ of $\pi(V)$ and closed subsets $Z \subseteq X$ such that $C_M(Z) \subseteq V^\circ$.

Let $u \in \mathcal{B}_M(M)$ be a hyperfunction on M. The corresponding microfunction $sp(u)$ is a section of $\mathcal{C}_M$ and its support $\operatorname{supp}(sp(u)) \subseteq T^*_M X$ is a closed conic subset.

Definition 5.2.4. The set $\operatorname{supp}(sp(u)) \subseteq T^*_M X$ is called the *singular support* of u and denoted by $SS(u)$.

Remark 5.2.5. The above exact sequence shows that $SS(u) \subseteq T^*_X X$ if and only if u is analytic and $\mathring{\pi}(SS(u))$ is the locus where u fails to be an analytic function. Hence we can think of $SS(u)$ as the set of directions in which u is not analytic.

There is a natural way to create hyperfunctions by taking boundary values of holomorphic functions defined on sufficiently nice open sets.

Definition 5.2.6. An an open subset $\Omega \subseteq X$ is called M-admissible if the following two conditions are satisfied:

1. $M \subseteq \overline{\Omega}$.

2. Ω is locally cohomological trivial (l.c.t.), i.e. there are natural isomorphisms

$$D'(\mathbb{C}_\Omega) = R\mathcal{H}om(\mathbb{C}_\Omega, \mathbb{C}_X) \cong \mathbb{C}_{\overline{\Omega}}$$
$$D'(\mathbb{C}_{\overline{\Omega}}) = R\mathcal{H}om(\mathbb{C}_{\overline{\Omega}}, \mathbb{C}_X) \cong \mathbb{C}_\Omega.$$

The first condition is clearly necessary for defining a "boundary" value, while the second ensures that the boundary is not too wild.

Proposition 5.2.7. $\Omega \subseteq X$ *is l.c.t. if the boundary* $\partial\Omega$ *is a* C^0 *submanifold of* X.

Proof. The required isomorphisms always hold for $x \in \Omega$ or $x \notin \overline{\Omega}$. For $x \in \partial\Omega = \overline{\Omega}\backslash\Omega$ we can find a cofinal system of contractible coordinate neighbourhoods (U, φ) of class C^0 such that

$$U\backslash\overline{\Omega} = \{x \in U \mid \varphi_1(x) < 0\}.$$

For these sets, the map $\Gamma(U, \mathbb{C}) \to \Gamma(U\backslash\Omega)$ is an isomorphism and the higher cohomology groups vanish. From the distinguished triangle of Exam. 2.1.10, we obtain $R\Gamma_{\overline{\Omega}}(U, \mathbb{C}) = 0$. Taking the colimit over all U as above shows $(R\Gamma_{\overline{\Omega}}\mathbb{C}_X)_x = 0$. The same argument applied to the complement Ω^c shows that $(R\Gamma_{\Omega^c}\mathbb{C}_X)_x = 0$ and thus the natural map $\mathbb{C}_{X,x} \to (R\Gamma_{\Omega}\mathbb{C}_X)_x$ is an isomorphism. □

Example 5.2.8. Suppose $\Omega \subseteq \mathbb{C}^n$ is a convex subset. Then so is its closure $\overline{\Omega}$ and the boundary $\partial\Omega$ is a C^0-submanifold. Hence Ω is l.c.t.

The l.c.t.-property is also stable under non-characteristic inverse images.

Proposition 5.2.9. *Let $f : Y \to X$ be a holomorphic map and $\Omega \subseteq X$ an l.c.t. subset. If f is non-characteristic for $SS(\mathbb{C}_\Omega)$, then $\overline{f^{-1}(\Omega)} = f^{-1}(\overline{\Omega})$ and the inverse image $f^{-1}(\Omega) \subseteq Y$ is also an l.c.t. subset.*

Proof. From Prop. 2.6.13 and Thm. 2.2.3 we get the isomorphisms

$$\begin{aligned}\mathbb{C}_{f^{-1}(\Omega)} \otimes \omega_{Y/X} &\cong f^{-1}\mathbb{C}_\Omega \otimes \omega_{Y/X} \cong f^!\mathbb{C}_\Omega \\ &\cong f^! R\mathcal{H}om(\mathbb{C}_{\overline{\Omega}}, \mathbb{C}_X) \cong R\mathcal{H}om(\mathbb{C}_{f^{-1}(\overline{\Omega})}, \omega_{Y/X})\end{aligned}$$

Tensoring with $\omega_{Y/X}^{-1}$ gives

$$R\mathcal{H}om(\mathbb{C}_{f^{-1}(\overline{\Omega})}, \mathbb{C}_Y) \cong \mathbb{C}_{f^{-1}(\Omega)}.$$

From Example 2.6.15 we also know that

$$SS(\mathbb{C}_{\overline{\Omega}}) = SS(R\mathcal{H}om(\mathbb{C}_\Omega, \mathbb{C}_X)) \subseteq SS(\mathbb{C}_\Omega)^a.$$

Hence $SS(\mathbb{C}_{\overline{\Omega}})$ is non-characteristic for f as well and the same argument shows

$$R\mathcal{H}om(\mathbb{C}_{f^{-1}(\Omega)}, \mathbb{C}_Y) \cong \mathbb{C}_{f^{-1}(\overline{\Omega})}.$$

Since

$$\operatorname{supp}(R\mathcal{H}om(\mathbb{C}_{f^{-1}(\Omega)}, \mathbb{C}_Y)) \subseteq \overline{f^{-1}(\Omega)} \subseteq f^{-1}(\overline{\Omega}) = \operatorname{supp}(\mathbb{C}_{f^{-1}(\overline{\Omega})}),$$

this implies $f^{-1}(\overline{\Omega}) = \overline{f^{-1}(\Omega)}$ and $f^{-1}(\Omega)$ is an l.c.t. subset. □

Example 5.2.10. Let $\Omega_1, \Omega_2 \subseteq X$ be two open subsets, such that

$$SS(\mathbb{C}_{\Omega_1}) \cap SS(\mathbb{C}_{\Omega_2})^a \subseteq T_X^*X.$$

Then the diagonal map $\Delta : X \to X \times X$ is non-characteristic for

$$SS(\mathbb{C}_{\Omega_1 \times \Omega_2}) \subseteq SS(\mathbb{C}_{\Omega_1}) \times SS(\mathbb{C}_{\Omega_2})$$

and the intersection $\Omega_1 \cap \Omega_2 = \Delta^{-1}(\Omega_1 \times \Omega_2)$ is again *l.c.t.*.

Now suppose $j : \Omega \hookrightarrow X$ is the inclusion of an M-admissible subset. The map $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ gives by duality a map

$$\omega_{M/X} = R\mathcal{H}om(\mathbb{C}_M, \mathbb{C}_X) \to R\mathcal{H}om(\mathbb{C}_{\overline{\Omega}}, \mathbb{C}_X) \cong \mathbb{C}_\Omega$$

and applying $R^0\mathcal{H}om(-, \mathcal{O}_X)$ yields the boundary value map

$$b_\Omega : j_*j^{-1}\mathcal{O}_X\big|_M \cong R\mathcal{H}om(\mathbb{C}_\Omega, \mathcal{O}_X)\big|_M \to R\mathcal{H}om(\omega_{M/X}, \mathcal{O}_X) \cong \mathcal{B}_M.$$

We can also consider b_Ω as a map $\mu hom(\mathbb{C}_\Omega, \mathcal{O}_X) \to \mu hom(\omega_{M/X}, \mathcal{O}_X)$.

Proposition 5.2.11. *If $u = b_\Omega(g)$ for $g \in \mathcal{O}_X(\Omega)$, then $SS(u) \subseteq SS(\mathbb{C}_\Omega) \cap T^*_M X$.*

Proof. Let $sp(g) \in \Gamma(T^*X, \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X))$ be the image of g under the isomorphism

$$R^0 \operatorname{Hom}(\mathbb{C}_\Omega, \mathcal{O}_X) \cong R\Gamma^0(T^*X, \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)).$$

Then

$$\operatorname{supp}(sp(g)) \subseteq \operatorname{supp}(\mu hom(\mathbb{C}_\Omega, \mathcal{O}_X)) \subseteq SS(\mathbb{C}_\Omega),$$

where the last estimate follows from Prop. 2.6.16. Then $sp(b_\Omega(g)) = b_\Omega(sp(g))$ and its support satisfies

$$\operatorname{supp}(b_\Omega(sp(g))) \subseteq \operatorname{supp}(sp(g)) \cap \operatorname{supp} \mu hom(\omega_{M/X}, \mathcal{O}_X) \subseteq SS(\mathbb{C}_\Omega) \cap T^*_M X.$$

□

Note that $\mathcal{B}_M$ is naturally a $\mathcal{D}_M = \mathcal{D}_X\big|_M$-module. If a hyperfunctions $u \in \mathcal{B}_M$ satisfies a large system of differential equations, then this gives another way to constrain its singular support.

Proposition 5.2.12. *Suppose N is a coherent $\mathcal{D}_X$-module and $u \in \mathcal{H}om_{\mathcal{D}_X}(N, \mathcal{B}_M)$ is a hyperfunction solution to the system of differential equations corresponding to N. Then*

$$SS(u) \subseteq Ch(N) \cap T^*_M X.$$

Proof. We can regard u has a global section of

$$\begin{aligned} R^0\mathcal{H}om_{\mathcal{D}_X}(N, R\mathcal{H}om(\omega_{M/X}, \mathcal{O}_X)) &\cong R^0\mathcal{H}om(\omega_{M/X}, R\mathcal{H}om_{\mathcal{D}_X}(N, \mathcal{O}_X)) \\ &\cong R^0\mathcal{H}om(\omega_{M/X}, Sol(N)). \end{aligned}$$

We have $SS(Sol(N)) = Ch(N)$ by Thm. 4.6.3 and thus the assertion follows from Prop. 2.6.16. □

The following construction will be our main source of hyperfunctions and distributions:

Proposition 5.2.13. *Let $f : X \to \mathbb{C}$ be a non-constant holomorphic map, such that $f(M) \subseteq \mathbb{R}$ and*

$$g : \Omega_+ = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\} \to \mathbb{C}$$

a holomorphic function in the upper half plane. Let

$$\Omega_f = \{x \in X \mid \operatorname{Im}(f(x)) > 0\}$$

and denote by $Crit(f|_M)$ the critical points of $f|_M$. Suppose g extends to a holomorphic function in a neighbourhood $W \subseteq \mathbb{C}$ of $f(Crit(f|_M))$. Then there is a unique hyperfunction

$$u := g(f(x) + i0) \in \mathcal{B}_M(M),$$

which is given by the boundary value $u = b_{\Omega_f}(f \circ g)$ outside $Crit(f)$ and its singular support satisfies

$$SS(u) \subseteq T^*_X X \cap \{(x,\xi) \in T^*_M X \mid f(x) \notin W \text{ and } \xi = -\lambda \operatorname{Im}(\mathrm{d}f_x) \text{ for some } \lambda \geq 0\}.$$

Proof. Let $U = M \backslash Crit(f)$. In a complex neighbourhood of U we can locally find holomorphic coordinates, such that $f(z) = z_1$. Hence Ω_f is admissible for U and the boundary value $b_\Omega(f)$ is well-defined there. It follows from the Cauchy-Riemann equations, that $\mathrm{d}\operatorname{Im} f = \operatorname{Im}(\mathrm{d}f)$. Hence Example 2.6.5 gives

$$SS(\mathbb{C}_\Omega) = \{(z, -\lambda \operatorname{Im}(\mathrm{d}f_z)) \in T^*X \mid \lambda \operatorname{Im} f(z) = 0, \lambda \geq 0, \operatorname{Im}(f(z)) \geq 0\}.$$

in a complex neighbourhood of U. Restricting to $T^*_U X$ gives

$$SS(b_{\Omega_f}(g \circ f)|_U) \subseteq \{(x, -\lambda \mathrm{d}\operatorname{Im}(\mathrm{d}_x)) \in T^*_U X \mid \lambda \operatorname{Im} f(z) = 0, \lambda \geq 0, \operatorname{Im}(f(z)) \geq 0\}$$

By hypothesis the function $g \circ f$ extends to a holomorphic function on the complex neighbourhood $f^{-1}(W)$ of $Crit(f|_M)$ and its boundary value agrees with $b_{\Omega_f}(g \circ f)$ on the overlap $U \cap V$. Hence we can glue the two sections to obtain a well-defined hyperfunction $u \in \mathcal{B}_M(M)$, with

$$SS(u) \subseteq T^*_X X \cup \{(x,\xi) \in T^*_U X \mid f(x) \notin W, \xi = -\lambda \operatorname{Im}(\mathrm{d}f_x) \text{ for some } \lambda \geq 0\}.$$

If $\tilde{u} \in \mathcal{B}_M(M)$ is another hyperfunction with the above properties then $u - \tilde{u}$ is an analytic function with support $Crit(f)$. Hence $u - \tilde{u} = 0$ by analytic continuation. $\square$

From now on, we will identify $T^*_M X$ with iT^*M by the map $i\mathrm{d}x \mapsto \mathrm{d}y$. Hence we can write the singular support of the previous example as

$$SS(g(f(x) + i0)) \subseteq iT^*_M M \cup \{(x, i\xi) \in iT^*M \mid f(x) \notin W, \xi = \lambda \mathrm{d}f(x) \text{ for some } \lambda \geq 0\}.$$

5.3 Distributions from boundary values

Let M be a real analytic manifold of dimension m and $i : M \hookrightarrow X$ a complexification. From Prop. 5.1.10, we obtain a homomorphism

$$\begin{aligned} Db_M^\vee &\to \mathcal{T}hom(\mathbb{C}_M, Db_X^{m,m}) \\ &\to (\mathcal{T}hom(\mathbb{C}_M, Db_X^{m,\bullet}), \overline{\partial})[m] \\ &\cong \mathcal{T}hom(\mathbb{C}_M, \Omega_X^m)[m]. \end{aligned}$$

By side-changing, we get the map

$$Db_M \to \mathcal{T}hom(\omega_{M/X}, \mathcal{O}_X).$$

This is in fact an isomorphism. More generally, we have the following.

Theorem 5.3.1 ([KS96, Thm 5.10][And94, Prop. 1.2.5])**.** *For $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_M)$ there is a natural isomorphism*

$$\mathcal{T}hom(i_*F, \Omega_X[\dim_{\mathbb{C}} X]) \cong i_*\mathcal{T}hom(F, Db_M^\vee)$$

For $F = D'(\mathbb{C}_M)$ this induces an isomorphism

$$\mathcal{T}hom(D'(\mathbb{C}_M), \mathcal{O}_X) \cong Db_M.$$

By Remark 5.1.14, we have a a natural map

$$Db_M \cong \mathcal{T}hom(\omega_{M/X}, \mathcal{O}_X) \to R\operatorname{Hom}(\omega_{M/X}, \mathcal{O}_X) \cong \mathcal{B}_M$$

Theorem 5.3.2 ([Sch70, Thm. 122])**.** *The map $Db_M \to \mathcal{B}_M$ is injective.*

Let $\Omega \subseteq X$ be a subanalytic open subset, which is admissible for $M \hookrightarrow X$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{T}hom(\mathbb{C}_\Omega, \mathcal{O}_X) & \xrightarrow{b_\Omega} & Db_M \\ \downarrow & & \downarrow \\ R\mathcal{H}om(\mathbb{C}_\Omega, \mathcal{O}_X) & \xrightarrow{b_\Omega} & \mathcal{B}_M. \end{array}$$

We denote by $\mathcal{T}\mathcal{O}_X(\Omega)$ the sections of $H^0(X, \mathcal{T}hom(\mathbb{C}_\Omega, \mathcal{O}_X))$. These are given by holomorphic functions $f \in \Gamma(\Omega, \mathcal{O}_X)$, which have moderate growth at the boundary of Ω or, equivalently, have a (non-unique) distributional extension to X. The boundary value map gives a distribution $b_\Omega(f) \in Db_M$. In this case, the boundary value has the following more concrete representation:

Theorem 5.3.3. *Let $\Omega \subseteq X$ be an M-admissible open subset and $f \in \mathcal{T}\mathcal{O}_X(\Omega)$. Suppose*

$$\gamma : [0, T) \times X \to X, \quad (t, x) \mapsto \gamma_t(x)$$

is a continuous map, such that:

1. γ_t *is holomorphic for all fixed* $t \in [0,T)$.
2. $\lim_{t\to 0^+} \gamma_t = \mathrm{id}_X$.
3. $\gamma_t(M) \subseteq \Omega$ *for* $t \in (0,T)$.

Then the boundary value map is given by

$$b_\Omega(f) = \lim_{t\to 0^+} (f \circ \gamma_t)|_M,$$

where the limit on the RHS is in the sense of distributions.

For the proof, we need a lemma. First let $x_0 \in \overline{\Omega}$ and, in a coordinate neighbourhood $U \subseteq X$ of x_0, consider the open ball

$$D_r(x_0) = \{z \in U \mid \|z - x_0\| < r\}.$$

Lemma 5.3.4. *There is* $r_0 > 0$ *such that the subset* $\Omega \cap D_r(x_0)$ *is l.c.t. for all* $0 < r < r_0$.

Proof. Let

$$\varphi : U \to \mathbb{R}, \quad \varphi(z) = \|z - x_0\|^2.$$

Then φ is proper and analytic. Prop. 2.7.9 applied with $F = \mathbb{C}_{\partial\Omega}$ shows that the image $\varphi(S)$ of the set

$$S = \{z \in U \mid -\mathrm{d}\varphi(z) \in SS(\mathbb{C}_{\partial\Omega})\}$$

is discrete. We can then find $r_0 > 0$ such that $\varphi^{-1}((0, r_0)) \cap S = \emptyset$. Since

$$SS(\mathbb{C}_{D_r(x_0)}) = \{(z, \lambda \mathrm{d}\varphi(z)) \in T^*U \mid \varphi(z) \le r, \lambda \ge 0, \lambda(\phi(z) - r) = 0\},$$

by Exam. 2.6.5 and

$$SS(\mathbb{C}_\Omega) = T^*_X X \cap \pi^{-1}(\Omega) \cup SS(\mathbb{C}_{\partial\Omega}),$$

we have $SS(\mathbb{C}_\Omega) \cap SS(\mathbb{C}_{D_r(x_0)})^a \subseteq T^*_X X$ for $0 < r < r_0$. Then the assertion follows from Exam. 5.2.10. ☐

Fix $r > 0$ such that $\Omega_0 = \Omega \cap D_r(x_0)$ is l.c.t. Let $B = D_r(x_0) \cap M$ be the corresponding ball in M and $\overline{B} \subseteq M$ its closure. Note that we have isomorphisms

$$\begin{aligned} D'(\mathbb{C}_{\overline{B}}) &\cong R\mathcal{H}om(\mathbb{C}_{\overline{B}}, \mathbb{C}_X) \cong R\mathcal{H}om(\mathbb{C}_{\overline{B}} \otimes \mathbb{C}_M, \mathbb{C}_X) \\ &\cong R\mathcal{H}om(\mathbb{C}_{\overline{B}}, R\mathcal{H}om(\mathbb{C}_M, \mathbb{C}_X)) \cong R\mathcal{H}om(\mathbb{C}_{\overline{B}}, or_M)[-m] \\ &\cong or_B[-m], \end{aligned}$$

where the last equality follows from Prop. 5.2.7.

As above, we have the maps $or_B[-m] \cong D'(\mathbb{C}_{\overline{B}}) \to D'(\mathbb{C}_{\overline{\Omega_0}}) \cong \mathbb{C}_{\Omega_0}$ and thus the boundary value map

$$b_{\Omega_0} : \mathcal{T}\mathcal{O}_X(\Omega_0) \to H^0(X, \mathcal{T}hom(D'(\mathbb{C}_{\overline{B}}), \mathcal{O}_X)) \cong \Gamma(M, \mathcal{T}hom(\mathbb{C}_B, Db_M)).$$

The space on the right consists of distributions $u \in Db_M(B)$, which have moderate growth at the boundary of B (cf. Exam. 5.1.5).

Remark 5.3.5. It follows from Thm. 5.1.19, that the boundary value map[1]

$$b_{\Omega_0} : \mathcal{T}\mathcal{O}_X(\Omega_0) \to \Gamma(M, \mathcal{T}hom(\mathbb{C}_B, Db_M))$$

is continuous with respect to the topologies constructed there.

Proof of Thm. 5.3.3. The claim is local in the following sense: Let $\phi \in \Gamma_c(M, C_M^{\infty,\vee})$ be a compactly supported smooth density. Then we must show the equality

$$\langle b_\Omega(f), \phi \rangle = \lim_{t \to 0^+} \int_M f(\gamma_t(x))\phi(x),$$

where $\langle -, - \rangle$ denotes the pairing between distributions and compactly supported densities. By using a partition of unity, we can even assume that the support of ϕ is arbitrarily small.

Working locally around a point $x_0 \in \overline{\Omega}$, we can choose B and Ω_0 as above and assume that $\operatorname{supp}(\phi)$ is contained in B.

Shrinking B further if necessary, we can also assume that $\gamma_t(B) \subseteq \Omega_0$ for $t > 0$ small enough.

The natural map $\mathbb{C}_{\Omega_0} \to \mathbb{C}_\Omega$ induces $\mathcal{T}hom(\mathbb{C}_\Omega, \mathcal{O}_X) \to \mathcal{T}hom(\mathbb{C}_{\Omega_0}, \mathcal{O}_X)$ and by functoriality of $\mathcal{T}hom(-, \mathcal{O}_X)$, we have $\langle b_\Omega(f), \phi \rangle = \langle b_{\Omega_0}(f), \phi \rangle$. On the other hand, the functions $f_t(z) = f(\gamma_t(z))$ are holomorphic in a neighbourhood of $\overline{B}$ and converge to f in the topology of $\mathcal{T}\mathcal{O}_X(\Omega_0)$. We then have the commutative diagram

$$\begin{tikzcd}
\mathcal{T}hom(X, \mathcal{O}_X)|_{\overline{B}} \arrow[r] \arrow[d] & \mathcal{T}hom(\Omega_0, \mathcal{O}_X)|_{\overline{B}} \arrow[d, "b_{\Omega_0}"] \\
\mathcal{A}_M|_{\overline{B}} \arrow[r] & \mathcal{T}hom(\mathbb{C}_B, Db_M),
\end{tikzcd}$$

By continuity of b_{Ω_0}, we can conclude that

$$\langle b_\Omega(f), \phi \rangle = \langle b_{\Omega_0}(\lim_{t \to 0^+} f_t), \phi \rangle = \lim_{t \to 0^+} \langle b_{\Omega_0}(f_t), \phi \rangle = \lim_{t \to 0^+} \int_M f(\gamma_t(x))\phi(x).$$

$\square$

Example 5.3.6. The classical version of this construction, which can be found in e.g. [Hör98, Thm. 3.1.15], is the following: Let $\Gamma \subseteq \mathbb{R}^n$ be an open, proper convex cone and consider the open subset

$$\Omega = \mathbb{R}^n \times i\Gamma \subseteq \mathbb{C}^n.$$

Convexity implies that Ω has a C^0-boundary and is thus *l.c.t.* Let us also assume that Γ and thus Ω is subanalytic. If $f \in \mathcal{O}_X(\Omega)$ has moderate growth on the boundary of Ω, then the boundary value $b_\Omega(f) \in \Gamma(\mathbb{R}^n, Db_{\mathbb{R}^n})$ is well-defined. Choose $y \in \Gamma$ and let $\gamma_t(z) = z + ity$. Then the hypothesis of the above proposition are satisfied and we get

$$b_\Omega(f) = \lim_{t \to 0^+} f \circ \gamma_t,$$

[1]The author thanks Pierre Schapira for pointing this out.

in the sense of distributions. In other words, if $\phi \in \Gamma_c(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$ is a compactly supported test function, then

$$\langle b_\Omega(f), \phi \mathrm{d}x \rangle = \lim_{t \to 0^+} \int_{\mathbb{R}^n} f(x + ity)\phi(x)\mathrm{d}x.$$

Example 5.3.7. Suppose $f : (X, M) \to (\mathbb{C}, \mathbb{R})$ and $g : \Omega_+ \to \mathbb{C}$ satisfy the hypothesis of Prop. 5.2.13 and assume additionally, that g has moderate growth on $\mathbb{R}$. Then the corresponding hyperfunction

$$u(x) = g(f(x) + i0)$$

is a distribution. We can alternatively describe u as the limit

$$u(x) = \lim_{t \to 0^+} g(f(\gamma_t(x))),$$

where $\gamma : [0, T) \times X \to X$ is a a continuous map as above, such that γ_t is holomorphic and $\operatorname{Im} f(\gamma_t(x)) > 0$ for $t > 0$.

Since the map $i : Db_M \to \mathcal{B}_M$ is injective, it is sensible to use $SS(i(u))$ as a measure of non-analyticity of a distribution u.

Definition 5.3.8. The singular support of a distribution $u \in Db_M(M)$ is the singular support of u as a hyperfunction:

$$SS(u) := SS(i(u)) \subseteq iT^*M$$

Remark 5.3.9. It is shown in [Bon77], that $SS(u)\backslash iT^*_M M$ agrees with Hörmander's analytic wave front set $WF_A(u)$ defined in e.g. [Hör98]. In particular, we have

$$WF(u) \subseteq SS(u)\backslash iT^*_M M,$$

where $WF(u)$ is the smooth wave front set.

5.4 Pullback and pushforward

Let $f : N \to M$ be an analytic map. We can extend f to a holomorphic map $f : Y \to X$ of suitable complexifications. The associated conormal bundle map

$$f^* : T^*_M X \cong iT^*M \to T^*_N Y \cong iT^*N$$

decomposes as

$$iT^*N \xleftarrow{f_d} N \times_M iT^*M \xrightarrow{f_\pi} iT^*M.$$

It is not always possible to define the pullback of a hyperfunction along f. Microlocalization gives a way to extend the usual pullback of analytic functions if "the problems come from different directions", i.e. when the singular support is non-characteristic for f.

Proposition 5.4.1. *Let $f : (Y,N) \to (X,M)$ be a morphism of pairs as above.*

1. *There is a natural morphism*

$$Rf_{d!}f_\pi^{-1}\mathcal{C}_M \to \mathcal{C}_N.$$

2. *If u is a hyperfunction on M and $SS(u)$ is non-characteristic for f, then the pull-back $f^*(u) \in \mathcal{B}_N(N)$ is a well-defined hyperfunction on N with*

$$SS(f^*u) \subseteq f_d(f_\pi^{-1}(SS(u))).$$

3. *Suppose $\Omega \subseteq X$ is admissible for M and f is non-characteristic for $SS(\mathbb{C}_\Omega)$. Then $f^{-1}(\Omega)$ is admissible for $N \subseteq Y$. If $u = b_\Omega(g)$ is the boundary value of $g \in R\mathcal{H}om(\mathbb{C}_\Omega, \mathcal{O}_X)$ then the pullback $f^*(u)$ is the boundary value:*

$$f^*(u) = b_{f^{-1}(\Omega)}(f^*g).$$

If additionally Ω is subanalytic and $g \in \mathcal{T}\mathcal{O}_X(\Omega)$, then $f^*g \in \mathcal{T}\mathcal{O}_Y(f^{-1}(\Omega))$ and $f^*(u) = b_{f^{-1}(\Omega)}(f^*g) \in Db_N$.

Proof. For $F_1, F_2 \in D^b(\mathbb{C}_X)$, Prop. 2.5.14 gives a natural map

$$Rf_{d!}f_\pi^{-1}\mu hom(F_1,F_2) \to \mu hom(f^!F_1, f^{-1}F_2 \otimes \omega_{Y/X}).$$

1. For $F_1 = \omega_{M/X}$ and $F_2 = \mathcal{O}_X$ we get

$$\begin{aligned} Rf_{d!}f_\pi^{-1}\mathcal{C}_N &= Rf_{d!}f_\pi^{-1}\mu hom(\omega_{M/X}, \mathcal{O}_X) \\ &\to \mu hom(f^!\omega_{M/X}, f^{-1}\mathcal{O}_X \otimes \omega_{Y/X}) \\ &= \mu hom(f^!\omega_{M/X} \otimes \omega_{Y/X}^{-1} \otimes \omega_{Y/X}, f^{-1}\mathcal{O}_X \otimes \omega_{Y/X}) \end{aligned}$$

   Using the isomorphism $\omega_{N/Y} \cong f^!\omega_{M/X} \otimes \omega_{Y/X}^{-1}$ and the structure morphism $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ gives the maps

$$\begin{aligned} \mu hom(\omega_{N/Y} \otimes \omega_{X/Y}, f^{-1}\mathcal{O}_X \otimes \omega_{Y/X}) &\cong \mu hom(\omega_{N/Y}, f^{-1}\mathcal{O}_X) \\ &\to \mu hom(\omega_{N/Y}, \mathcal{O}_Y) = \mathcal{C}_N \end{aligned}$$

2. That f is non-characteristic for u means that $f^{-1}sp(u)$ is a section of $R^0 f_{d!}f^{-1}\mathcal{C}_M$. The above map then gives a section $f^*u \in \Gamma(T_N^*Y, \mathcal{C}_N) \cong \Gamma(N, \mathcal{B}_N)$ with support contained in $f_d f_\pi^{-1}(SS(u))$.

3. By Prop. 5.2.7, $f^{-1}(\Omega)$ is l.c.t. and $f^{-1}(\overline{\Omega}) = \overline{f^{-1}(\Omega)}$. It follows that $N \subseteq \overline{f^{-1}(\Omega)}$ and $f^{-1}(\Omega)$ is admissible for N. We have a morphism

$$\begin{aligned} Rf_{d!}f_\pi^{-1}\mu hom(\mathbb{C}_\Omega, \mathcal{O}_X) &\to \mu hom(f^!\mathbb{C}_\Omega, f^{-1}\mathcal{O}_X \otimes \omega_{Y/X}) \\ &\cong \mu hom(\mathbb{C}_{f^{-1}(\Omega)}, f^{-1}\mathcal{O}_X). \end{aligned}$$

Again combining with $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ gives the map

$$Rf_{d!}f_\pi^{-1}\mu hom(\mathbb{C}_\Omega, \mathcal{O}_X) \to \mu hom(\mathbb{C}_{f^{-1}(\Omega)}, \mathcal{O}_Y),$$

fitting into the commutative diagram

$$\begin{tikzcd}
Rf_{d!}f_\pi^{-1}\mu hom(\mathbb{C}_\Omega, \mathcal{O}_X) \arrow[r] \arrow[d, "Rf_{d!}f_\pi^{-1}b_\Omega"] & \mu hom(\mathbb{C}_{f^{-1}(\Omega)}, \mathcal{O}_Y) \arrow[d, "b_{f^{-1}(\Omega)}"] \\
Rf_{d!}f_\pi^{-1}\mathcal{C}_M \arrow[r] & \mathcal{C}_N
\end{tikzcd}$$

This shows $f^*(u) = b_{f^{-1}(\Omega)}(f^*g)$. If $g \in \mathcal{T}\mathcal{O}_X(\Omega) = \mathcal{T}hom(\mathbb{C}_\Omega, \mathcal{O}_X)$ then

$$f^*g \in \mathcal{T}hom(f^{-1}\mathbb{C}_\Omega, \mathcal{O}_Y) \cong \mathcal{T}hom(\mathbb{C}_{f^{-1}(\Omega)}, \mathcal{O}_Y) = \mathcal{T}\mathcal{O}_Y(f^{-1}(\Omega))$$

by Thm. 5.1.18. Hence the boundary value $f^*(u) = b_{f^{-1}(\Omega)}(f^*g)$ is a distribution.

$\square$

In the C^∞-setting, Hörmander [Hör98] defines an alternative pullback map. If $u \in Db_M(M)$ is a distribution, such that the f is non-characteristic for the C^∞ wavefront set $WF(u)$, then the pullback $f_H^*u \in Db_N(N)$ is well-defined. Moreover this map is continuous in the following sense ([Hör98, Thm. 8.2.4]): If $u = \lim_{n\to\infty} u_n$ is expressed as the limit of smooth functions $u_n \in C_M^\infty(M)$, then

$$f_H^*(u) = \lim_{n\to\infty} f^*u_n.$$

Since the singular support agrees with the analytic wavefront set, we have in particular $WF(u) \subseteq SS(u)$. Hence both pullback constructions are well-defined if f is non-characteristic for $SS(u)$.

Proposition 5.4.2 ([Bon77])**.** *If $u \in Db_M(M)$ and f is non-characteristic for $SS(u)$, then the pullback $f_H^*(u) \in Db_N$ agrees with the pullback $f^*(u)$ constructed above.*

Example 5.4.3. Suppose $f : N \to M$ is an analytic diffeomorphism. Then $f^*(u)$ is always well-defined since f is submersive. The map on distributions $f^* : Db_M(M) \to Db_N(N)$ is dual to the pullback map $(f^{-1})^* : \Gamma_c(M, C_M^{\infty,\vee}) \to \Gamma_c(N, C_N^{\infty,\vee})$. For smooth functions $g \in C_M^\infty$ this follows directly from the usual coordinate change formula: If $\phi \in \Gamma_c(M, C_M^{\infty,\vee})$, then

$$\langle f^*g, \phi\rangle = \int_N f^*g \cdot \phi = \int_{f(N)} g \cdot (f^{-1})^*\phi = \langle g, (f^{-1})^*\phi\rangle.$$

The general case follows by continuity of the pullback, since smooth functions are dense in $Db_M(M)$.

As an application of the preceding results, we can define the product of hyperfunctions and distributions under suitable non-characteristic hypothesis. First let us show that there is an external product map

Proposition 5.4.4. *Let M_1, M_2 be real analytic manifolds and X_1, X_2 corresponding complexifications. There are natural maps*

$$\begin{aligned}\boxtimes &: \mathcal{C}_{M_1} \boxtimes \mathcal{C}_{M_2} \to \mathcal{C}_{M_1 \times M_2} \\ \boxtimes &: \mathcal{B}_{M_1} \boxtimes \mathcal{B}_{M_2} \to \mathcal{B}_{M_1 \times M_2}.\end{aligned}$$

If $u_1(x) \in \mathcal{B}_{M_1}(M_1)$ and $u_2(y) \in \mathcal{B}_{M_2}(M_2)$ then their external product $u_1(x) \cdot u_2(y) := u_1 \boxtimes u_2$ satisfies

$$SS(u_1(x) \cdot u_2(y)) \subseteq SS(u_1(x)) \times SS(u_2(y)).$$

Proof. Applying Prop. 2.5.9 to $\mathcal{C}_{M_i} \cong \mu_{M_1}(\mathcal{O}_{X_i}) \otimes \omega_{M_i/X_i}$ and using the natural map $\mathcal{O}_{X_1} \boxtimes \mathcal{O}_{X_2} \to \mathcal{O}_{X_1 \times X_2}$ gives the map

$$\begin{aligned}\mathcal{C}_{M_1} \boxtimes \mathcal{C}_{M_2} &\cong \mu_{M_1}(\mathcal{O}_{X_1}) \otimes \omega_{M_1/X_1} \boxtimes \mu_{M_2}(\mathcal{O}_{X_2}) \otimes \omega_{M_2/X_2} \\ &\to \mu_{M_1 \times M_2}(\mathcal{O}_{X_1} \boxtimes \mathcal{O}_{X_2}) \otimes \omega_{M_1 \times M_2/X_1 \times X_2} \\ &\to \mu_{M_1 \times M_2}(\mathcal{O}_{X_1 \times X_2}) \otimes \omega_{M_1 \times M_2/X_1 \times X_2} \\ &\cong \mathcal{C}_{M_1 \times M_2}.\end{aligned}$$

The map on hyperfunctions is obtained by the functor $R\pi_{12*}$, where

$$\pi_{12} : T^*_{M_1} \times T^*_{M_2} X_1 \cong T^*_{M_1 \times M_2} X_1 \times X_2 \to M_1 \times M_2,$$

is the natural projection. By constructions we have $\mathrm{supp}(\mu_1 \boxtimes \mu_2) \subseteq \mathrm{supp}(\mu_1) \times \mathrm{supp}(\mu_2)$ for two microfunctions $\mu_i \in \mathcal{C}_{M_i}$. For two hyperfunctions $u_i \in \mathcal{B}_{M_i}$, we then have

$$SS(u_1 \boxtimes u_2) = \mathrm{supp}(sp(u_1) \boxtimes sp(u_2)) \subseteq SS(u_1) \times SS(u_2).$$

□

Remark 5.4.5. We have a commutative diagram

$$\begin{array}{ccc} Db_{M_1} \times Db_{M_2} & \longrightarrow & Db_{M_1 \times M_2} \\ \downarrow & & \downarrow \\ \mathcal{B}_{M_1} \times \mathcal{B}_{M_2} & \longrightarrow & \mathcal{B}_{M_1 \times M_2} \end{array}$$

where the first horizontal map is a special case of Thm. 5.1.17 under the identification $Db_M \cong \mathcal{T}hom(\omega_{M/X}, \mathcal{O}_X)$.

Now let $M_1 = M_2 = M$ and $\Delta : M \to M \times M$ be the diagonal. The above proposition gives a pullback map of microfunctions

$$R\Delta_{d!}\Delta_\pi^{-1}\mathcal{C}_M \boxtimes \mathcal{C}_M \to R\Delta_{d!}\Delta_\pi^{-1}\mathcal{C}_{M \times M} \to \mathcal{C}_M$$

Corollary 5.4.6. *Let $u_1, u_2 \in \mathcal{B}_M$ be hyperfunctions, such that*

$$SS(u_1) \cap SS(u_2)^a \subseteq T^*_X X.$$

Then the product $u_1 \cdot u_2$ is well-defined and given as the pullback

$$u_1 \cdot u_2 = \Delta^*(u_1 \boxtimes u_2).$$

Its singular support satisfies

$$\begin{aligned} SS(u_1 \cdot u_2) &\subseteq \{\xi_1 + \xi_2 \mid \xi_1 \in SS(u_1), \xi_2 \in SS(u_2), \pi(\xi_1) = \pi(\xi_2)\} \\ &:= SS(u_1) + SS(u_2). \end{aligned}$$

Proof. The pullback map

$$\Delta_d : M \times_{M \times M} T^*_{M \times M} X \times X \to T^*_M X$$

is given on covectors $(\xi_x, \nu_x) \in (T^*_{M\times M} X \times X)_x$ by

$$\Delta_d(\xi_x, \nu_x) = \xi_x + \nu_x.$$

Hence the hypothesis imply that

$$\ker \Delta_d \cap \Delta_\pi^{-1}(SS(u_1) \times SS(u_2)) \subseteq T^*_X X,$$

i.e. Δ is non-characteristic for $u_1 \boxtimes u_2$ and the above product is well-defined. From Prop. 5.4.1 and 5.4.4 we have

$$SS(u_1 \cdot u_2) \subseteq \Delta_d(\Delta_\pi^{-1}(SS(u_1) \times SS(u_2))) = SS(u_1) + SS(u_2).$$

□

Example 5.4.7. Suppose Ω_1, Ω_2 are admissible subsets such that

$$SS(\mathbb{C}_{\Omega_1}) \cap SS(\mathbb{C}_{\Omega_2})^a \subseteq T^*_X X.$$

Exam. 5.2.10 shows that the intersection

$$\Omega_1 \cap \Omega_2 = \Delta^{-1}(\Omega_1 \times \Omega_2)$$

is again admissible and the product of $g_i \in \Gamma(\Omega_i, \mathcal{O}_X)$ can be defined as

$$b_{\Omega_1}(g_1) \cdot b_{\Omega_2}(g_2) = b_{\Omega_1 \cap \Omega_2}(g_1 \cdot g_2)$$

Remark 5.4.8. Suppose $u_1, u_2 \in Db_M$ are distributions and $u_1 = b_\Omega(g)$ is a boundary value of an admissible open subset, such that $SS(\mathbb{C}_\Omega) \cap SS(u_2)^a \subseteq T^*_X X$. Assume that $\gamma : [0, T) \times X \to X$ satisfies the conditions of Thm. 5.3.3 for Ω. Since the pullback is continuous by Prop. 5.4.2 we can compute the product as the limit

$$u_1(x) \cdot u_2(x) = \lim_{t \to 0^+} g(\gamma_t(x)) \cdot u_2(x).$$

Now let us try to define the pushforward of hyperfunction. In case f is a submersion, this is essentially integration along the fibers.

Let $\omega_Y = \Omega_Y^n$ be the sheaf of holomorphic forms of highest degree on Y and

$$\mathcal{V}_N = \Omega_Y^n\big|_N \otimes or_N$$

be the sheaf of analytic densities on N.

Proposition 5.4.9. *There is a pushforward morphism of microfunctions*

$$Rf_{\pi!}f_d^{-1}(\mathcal{C}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N) \to \mathcal{C}_M \otimes_{\mathcal{A}_M} \mathcal{V}_M.$$

Proof. We have the isomorphism

$$\begin{aligned}\mathcal{C}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N &= \mu_N(\mathcal{O}_Y) \otimes_{\mathcal{A}_N} \Omega_Y^n|_N \otimes or_{N/Y} \otimes or_N[n] \\ &\cong \mu_N(\Omega_Y^n)[n]\end{aligned}$$

Combining the morphism

$$Rf_{\pi!}f_d^{-1}(\mu_N(\Omega_Y^n[n])) \to \mu_M(Rf_!\Omega_Y^n[n])$$

from Prop. 2.5.9 with the integration morphism $Rf_!\Omega_Y^n[n] \to \Omega_X^m[m]$ gives the map

$$\begin{aligned}Rf_{\pi!}f_d^{-1}(\mathcal{C}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N) &\cong Rf_{\pi!}f_d^{-1}(\mu_N(\Omega_Y^n[n])) \\ &\to \mu_M(\Omega_X^m[m]) \cong \mathcal{C}_M \otimes_{\mathcal{A}_M} \mathcal{V}_M.\end{aligned}$$

□

Let

$$\mathcal{B}_N^\vee = \mathcal{B}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N,$$

be the sheaf of hyperfunction densities.

Corollary 5.4.10. *There is a natural morphism*

$$\begin{aligned}f_!(\mathcal{B}_N^\vee) &\to \mathcal{B}_M^\vee \\ u &\mapsto \int_f u,\end{aligned}$$

such that

$$SS\left(\int_f u\right) \subseteq f_\pi(f_d^{-1}(SS(u))).$$

Proof. Let $u \in f_!\mathcal{B}_N^\vee$. The image of $sp(u) \in \mathcal{C}_N^\vee$ under the map

$$\Gamma(iT^*N, \mathcal{C}_N^\vee) \to \Gamma(N \times_M iT^*M, f_d^{-1}\mathcal{C}_N^\vee)$$

has f_π proper support, so its image under the map $R^0f_{\pi!}f_d^{-1}\mathcal{C}_N^\vee \to \mathcal{C}_M^\vee$ is well-defined and gives a section

$$\int_f u \in \Gamma(iT^*M, \mathcal{C}_M^\vee) \cong \Gamma(M, \mathcal{B}_M^\vee).$$

By construction, its support is contained in $f_\pi f_d^{-1}(\operatorname{supp}(sp(u))) = f_\pi f_d^{-1}(SS(u))$. □

Remark 5.4.11. As in Remark 5.1.9, one can show that the above map factors through the natural map $f_!(\mathcal{B}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N) \to \underline{f}_!(\mathcal{B}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N)$. We have a commutative diagram

$$\begin{tikzcd}
f_!(Db_N^\vee) \arrow[r] \arrow[d] & \underline{f}_!(Db_N^\vee) \arrow[r] \arrow[d] & Db_M^\vee \arrow[d] \\
f_!(\mathcal{B}_N^\vee) \arrow[r] & \underline{f}_!(\mathcal{B}_N^\vee) \arrow[r] & \mathcal{B}_M^\vee .
\end{tikzcd}$$

Example 5.4.12. Let $f : N = M \times P \to M$ be a projection and

$$\mathcal{V}_{N/M} := \mathcal{V}_N \otimes_{\mathcal{A}_N} f^*\mathcal{V}_M$$

be the sheaf of relative densities. Tensoring the above morphism with $\mathcal{V}_M^{-1}$ gives the morphism

$$\begin{aligned} f_!(\mathcal{B}_N \otimes_{\mathcal{A}_N} \mathcal{V}_{N/M}) &\to \mathcal{B}_N, \\ u &\mapsto \int_P u. \end{aligned}$$

If (x, ξ) and (y, ν) are local coordinates of T^*P and T^*M, then $N \times_M T^*M$ can be described as

$$N \times_M T^*M = \{(x, y, \xi, \nu) \in T^*N \mid \xi = 0\}.$$

It follows that

$$SS\left(\int_P u\right) \subseteq f_\pi(f_d^{-1}(SS(u))) = \{(y, i\nu) \in iT^*M \mid \exists x \in P : (x, y, 0, i\nu) \in SS(u)\}.$$

5.5 Examples

Let us collect some examples of distributions by boundary values. We refer to [KKK86] and [Hör98] for similar arguments and further details.

Let $M = \mathbb{R} \subseteq X = \mathbb{C}$ and $\Omega_\pm \subseteq \mathbb{C}$ be the upper/lower half planes. For $f \in \mathcal{TO}_X(\Omega_\pm)$, we denote the corresponding distributions by

$$b_{\Omega_\pm}(f) = f(x \pm i0) \in Db_M(M).$$

Example 5.5.1. For $\lambda \in \mathbb{C}$, consider the hyperfunction $(x + i0)^\lambda = b_{\Omega_+}(z^\lambda)$, where $z^\lambda = e^{\lambda \log(z)}$ denotes the branch which is real-valued on $(0, \infty)$. This is a distribution, since the function z^λ is tempered at the boundary of Ω_+. For $\lambda \notin \mathbb{N}$, we have

$$SS((x + i0)^\lambda) = \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \geq 0, \alpha x = 0\}.$$

The inclusion $\subseteq$ follows directly from Prop. 5.2.13. If the inclusion were strict, then $(x + i0)^\lambda$ would be a real analytic function on $\mathbb{R}$, which is only possible if $\lambda \in \mathbb{N}$.

Example 5.5.2. The same argument shows that $\log(x+i0)$ is a distribution with singular support

$$SS(\log(x+i0)) = \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \geq 0, \alpha x = 0\}.$$

Example 5.5.3. For $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > -1$, the function $|x|^\lambda : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ is locally integrable near $0 \in \mathbb{R}$ and thus defines a distribution. We can extend $|x|^\lambda$ to a distribution-valued meromorphic function with simple poles at $\lambda \in \mathbb{Z}_{\leq -1} = \{-1, -2, \ldots\}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}|x|^\lambda = \mathrm{sgn}(x)\lambda|x|^{\lambda-1},$$

we have the equality

$$|x|^{\lambda-m} = \left(\prod_{k=0}^{m-1} \frac{\mathrm{sgn}(x)}{\lambda - k}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m |x|^\lambda.$$

This is an equality of distributions for $\mathrm{Re}(\lambda) > m-1$ and by analytic continuation, we can define the left hand side by the right hand side for $\mathrm{Re}(\lambda) < -1-m$. Since $|x|^\lambda$ is analytic on $\mathbb{R}\backslash\{0\}$, we have the estimate

$$SS(|x|^\lambda) \subseteq \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \in \mathbb{R}, x\alpha = 0\}.$$

Example 5.5.4. Let

$$\chi_+(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -1$, χ_+^λ is locally integrable and thus defines a distribution. As in the previous example, we can construct a meromorphic extension with simple poles along $\lambda \in \mathbb{Z}_{\leq -1}$, which we still denote by χ_+^λ. Following [KKK86, Example 2.4.2], we can show that it coincides with the boundary value distribution

$$\tilde{\chi}_+^\lambda = \frac{1}{e^{-\pi i\lambda} - e^{\pi i\lambda}}(e^{-\pi i\lambda}(x+i0)^\lambda - e^{\pi i\lambda}(x-i0)^\lambda),$$

for $\lambda \notin \mathbb{Z}$. By Thm. 5.3.3, we can write

$$\tilde{\chi}_+^\lambda(x) = \lim_{\epsilon\to 0^+} \frac{1}{e^{-\pi i\lambda} - e^{\pi i\lambda}}(e^{-\pi i\lambda}(x+i\epsilon)^\lambda - e^{\pi i\lambda}(x-i\epsilon)^\lambda).$$

For the principal branch of $z \mapsto z^\lambda$, we have

$$\lim_{\epsilon\to 0^+}(x+i\epsilon)^\lambda = x^\lambda = \lim_{\epsilon\to 0^+}(x-i\epsilon)^\lambda$$

for $x > 0$ and

$$\lim_{\epsilon\to 0^+}(x+i\epsilon)^\lambda = e^{i\pi\lambda}|x|^\lambda$$
$$\lim_{\epsilon\to 0^+}(x-i\epsilon)^\lambda = e^{-i\pi\lambda}|x|^\lambda$$

for $x < 0$. This shows that $\tilde{\chi}_+^\lambda$ agrees with the locally integrable function χ_+^λ for $\mathrm{Re}(\lambda) > -1, \lambda \notin \mathbb{N}$. Since it is also clearly meromorphic in λ, we must have $\chi_+^\lambda = \tilde{\chi}_+^\lambda$.

Since it vanishes on $(-\infty, 0)$ and is analytic in $(0, \infty)$, its singular support satisfies

$$SS(\chi_+^\lambda) \subseteq \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \in \mathbb{R}, x \in \mathbb{R}_{\geq 0}, \alpha x = 0\}.$$

Example 5.5.5. For $\lambda = 0$, we get $\chi_+^0 = \Theta(x)$, the classical Heaviside step function. An analogous argument gives the boundary value representation

$$\Theta(x) = \frac{\log(-x + i0) - \log(-x - i0)}{2\pi i},$$

and the singular support still satisfies the estimate

$$SS(\Theta(x)) \subseteq \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \in \mathbb{R}, x \in \mathbb{R}_{\geq 0}, \alpha x = 0\}.$$

Example 5.5.6. The derivative of $\Theta(x)$ is the Dirac delta $\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Theta(x)$. Its support is concentrated at $0 \in \mathbb{R}$, since $\Theta(x)$ is locally constant in $\mathbb{R}\backslash\{0\}$. Since $\frac{\mathrm{d}\log(z)}{\mathrm{d}z} = \frac{1}{z}$, we get

$$\delta(x) = \frac{1}{-2\pi i}\left(\frac{1}{x + i0} - \frac{1}{x - i0}\right)$$

It has support in $\{0\} \subseteq \mathbb{R}$, so we get

$$SS(\delta(x)) \subseteq \{(x, i\alpha \mathrm{d}x) \in iT^*\mathbb{R} \mid \alpha \in \mathbb{R}, x = 0\} = iT_0^*\mathbb{R}.$$

Let us return to the general case of a real-analytic manifold M, and suppose $M \subseteq X$ is a complexification.

Example 5.5.7. Suppose $f : X \to \mathbb{C}$ is a holomorphic function, which is real-valued on M and has 0 as a regular value. Applying Prop. 5.2.13 and Prop. 5.4.1 gives the following distributions:

$$\begin{aligned}
(f + i0)^\lambda &:= f^*((x + i0)^\lambda)\\
\chi_+^\lambda(f) &:= f^*(\chi_+^\lambda(x))\\
\Theta(f) &:= f^*(\Theta(x))\\
\delta(f) &:= f^*(\delta(x)).
\end{aligned}$$

Their singular support can be estimated as

$$\begin{aligned}
SS((f + i0)^\lambda) &\subseteq \{(x, i\alpha \mathrm{d}f(x)) \in iT^*M \mid \alpha \in \mathbb{R}_{\geq 0}, \alpha f(x) = 0\}\\
SS(\chi_+^\lambda(f)) &\subseteq \{(x, i\alpha \mathrm{d}f(x)) \in iT^*M \mid \alpha \in \mathbb{R}, f(x) \geq 0, \alpha f(x) = 0\}\\
SS(\Theta(f)) &\subseteq \{(x, i\alpha \mathrm{d}f(x)) \in iT^*M \mid \alpha \in \mathbb{R}, f(x) \geq 0, \alpha f(x) = 0\}\\
SS(\delta(f)) &\subseteq \{(x, i\alpha \mathrm{d}f(x)) \in iT^*M \mid \alpha \in \mathbb{R}, f(x) = 0\}.
\end{aligned}$$

Proposition 5.5.8. *Suppose $G = (g_1, \ldots, g_r) : M \to \mathbb{R}^r$ is an analytic functions, such that the divisor $D = \bigcup_{i=1}^r V(g_i)$ has simple normal crossings. Then the distribution*

$$u(\lambda) = \prod_{i=1}^r |g_i|^{\lambda_i} = G^*(\prod_{i=1}^r |x_i|^{\lambda_i})$$

is well-defined and meromorphic in λ, and satisfies

$$SS(u(\lambda)) \subseteq \bigcup_{I\subseteq\{1,\ldots,r\}} iT^*_{D^I} M,$$

where

$$D_I = \bigcup_{i\in I} V(g_i).$$

Proof. The external product $\prod_i |x_i|^{\lambda_i} \in Db_{\mathbb{R}^r}(\mathbb{R}^r)$ is well-defined and satisfies

$$\begin{aligned} SS\left(\prod_{i=1}^r |x_i|^{\lambda_i}\right) &\subseteq \{(x, i\sum_{i=1}^r \xi_i \mathrm{d}x_i) \in iT^*\mathbb{R}^r \mid \xi_i \in \mathbb{R}^r, \xi_i x_i = 0\} \\ &= \bigcup_{I\subseteq\{1,\ldots,r\}} iT^*_{\tilde{D}^I}\mathbb{R}^r, \end{aligned}$$

where $\tilde{D}_I = \bigcup_{i\in I} V(x_i)$. Then the pullback $G^*(\prod |x_i|^{\lambda_i})$ is well-defined if the functions $G_I = (g_i)_{i\in I} : M \to \mathbb{R}^{|I|}$ have 0 as regular value for all $I \subseteq \{1, \ldots, r\}$, i.e. if D is a simple normal crossing divisor. The estimate of its singular support is then immediate from Prop. 5.4.1. ☐

Proposition 5.5.9. *Let $F = (f_1, \ldots, f_d) : M \to \mathbb{R}^d$ be an analytic function, such that $0 \in \mathbb{R}^D$ is not a critical value, i.e. the intersection $N = \bigcap_i V(f_i)$ is an analytic submanifold of codimension d. Then the product*

$$\delta^d(F(x)) := \prod_{i=1}^d \delta(f_i(x))$$

is well-defined and

$$SS(\delta^d(F(x))) \subseteq \{(x, i\sum_{i=1}^d \alpha_i \mathrm{d}f_i(x)) \in iT^*M \mid \alpha_i \in \mathbb{R}, F(x) = 0\}.$$

Proof. The external product $\prod_{i=1}^d \delta(x_i) \in Db_{\mathbb{R}^D}(\mathbb{R}^D)$ is well-defined and its singular support satisfies

$$SS(\prod_{i=1}^d \delta(x_i)) \subseteq \{(0, i\sum_{j=1}^d \xi_j \mathrm{d}x_j) \in iT^*\mathbb{R}^d \mid \xi_j \in \mathbb{R}\} = iT^*_0\mathbb{R}^D.$$

Then $F^*(\xi) = 0$ for $\xi \in T^*_0\mathbb{R}^D$ implies $\xi = 0$, since $\mathrm{d}F$ is surjective, and the result follows from Prop. 5.4.1. ☐

Let us give a more concrete interpretation of the distribution $\delta^d(F)$.

Proposition 5.5.10. *Let $F : M \to \mathbb{R}^d$ be as above and suppose $\omega \in C_M^{\infty,n}(U)$ is a smooth n-form. Then we can find find an $(n-d)$-form $\tilde{\omega} \in C_M^{\infty,n-d}(U)$, such that*

$$\omega = \mathrm{d}F \wedge \tilde{\omega} := \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d \wedge \tilde{\omega}.$$

The restriction $\omega/\mathrm{d}F := \tilde{\omega}|_N$ is independent of the choice of $\tilde{\omega}$. The map $\omega \mapsto \omega/\mathrm{d}F$ induces a map $C_M^{\infty,\vee}|_N \to C_N^{\infty,\vee}$, such that for $\phi \in \Gamma_c(M, C_M^{\infty,\vee})$:

$$\langle \delta^d(F(x)), \phi \rangle = \int_N \phi/\mathrm{d}F$$

If M is oriented, then $/\mathrm{d}F$ induces an orientation of N and the product

$$\delta^d(F)\mathrm{d}F = \prod_{i=1} \delta(f_i)\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d$$

agrees with the current χ_N of the oriented submanifold N (see Exam. 2.3.3).

Proof. Locally, we can choose coordinates $(x_1, \ldots, x_n)$ such that $f_i(x) = x_i$. Then ω can be written as $\omega = w\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$, with $w \in C^\infty(M)$. Hence we can set

$$\tilde{\omega} = w\mathrm{d}x_{d+1} \wedge \ldots \wedge \ldots \mathrm{d}x_n.$$

If $\overline{\omega}$ is another differential form satisfying $\mathrm{d}F \wedge \overline{\omega} = \omega$, then

$$\overline{\omega}|_N = w|_N \mathrm{d}x_{d+1} \wedge \ldots \wedge \mathrm{d}x_n = \tilde{\omega}|_N.$$

Hence $\omega/\mathrm{d}F$ is well-defined and we obtain the map

$$C_M^{\infty,n}|_N \to C_N^{\infty,n-d}, \quad \omega \mapsto \omega/\mathrm{d}F.$$

If ω is a volume form on U, defining an orientation of U, then $\omega/\mathrm{d}N$ is a volume form on N. Thus an orientation on U induces an orientation on N through $/\mathrm{d}F$ and we obtain the isomorphism $or_M|_N \cong or_N$. This gives the map

$$/\mathrm{d}F : C_M^{\infty,\vee}|_N = C_M^{\infty,n}|_N \otimes or_M|_N \to C_N^{\infty,n-d} \otimes or_N = C_N^{\infty,\vee}.$$

Let $\phi \in \Gamma_c(M, C_M^{\infty,\vee})$. By using a partition of unity, we can assume that the support of ϕ is contained in a coordinate neighbourhood (V, x) as above. Then we can write $\phi = \phi_0|\mathrm{d}x|$ for some smooth function $\phi_0 \in C_M^\infty(U)$ and the value of $\delta^d(F)$ is given by

$$\langle \delta^d(F), \phi \rangle = \langle \prod_{i=1}^d \delta(x_i), \phi \rangle = \int_N \phi_0|_N \mathrm{d}x_{d+1} \ldots \mathrm{d}x_n. = \int_N \phi/\mathrm{d}F.$$

Now suppose M is oriented and N is given the orientation induced by the isomorphism $or_M|_N \cong or_N$. For $\phi \in \Gamma_c(M, C_M^{\infty,n-d})$ we can again assume that the support of ϕ is contained in V and we get

$$\begin{aligned}\langle \delta^d(F)\mathrm{d}F, \phi\rangle &= \langle \delta^d(F), \mathrm{d}f_1 \wedge \ldots \mathrm{d}f_d \wedge \phi\rangle \\ &= \int_N (\mathrm{d}f_1 \wedge \ldots \mathrm{d}f_d \wedge \phi)/\mathrm{d}F \\ &= \int_N \phi|_N = \langle \delta_N, \phi\rangle.\end{aligned}$$

□

Corollary 5.5.11. *If $G : U \to U$ is a diffeomorphism of a neighbourhood of $0 \in \mathbb{R}^d$ such that $G(0) = 0$, then*

$$\delta^d(G \circ F) = |\det(\mathrm{d}G(0))|^{-1}\delta^d(F).$$

Proof. The composition $G \circ F$ defines the same submanifold as F. Suppose first, that G is an oriented diffeomorphism. Then the orientations of N induced by $/\mathrm{d}(G \circ F)$ and $/\mathrm{d}F$ are the same and we have

$$\begin{aligned}\delta_N = \prod_{i=1}^{d} \delta(f_i)\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d &= \prod_{i=1}^{d} \delta((G \circ F)_i)\mathrm{d}(G \circ F)_1 \wedge \ldots \wedge \mathrm{d}(G \circ F)_d \\ &= \prod_{i=1}^{d} \delta((G \circ F)_i)\det(\mathrm{d}G(0))\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d\end{aligned}$$

i.e.

$$\delta^d(G \circ F) = \det(\mathrm{d}G(0))^{-1}\delta^d(F).$$

On the other hand, if G reverses orientation, i.e. $\det(\mathrm{d}G) < 0$, then N acquires the opposite orientation and we have

$$\prod_{i=1}^{d} \delta(f_i)\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d = -\prod_{i=1}^{d} \delta((G \circ F)_i)\det(\mathrm{d}G(0))\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_d.$$

Hence

$$\delta^d(G \circ F) = |\det(\mathrm{d}G(0))|^{-1}\delta^d(F).$$

□

Example 5.5.12. Suppose $G = \mathrm{diag}(t_1, \ldots, t_d)$. Then

$$\delta^d(G \circ F) = \prod_{i=1}^{d} \delta(t_i f_i) = \prod_{i=1}^{d} |t_i|^{-1}\delta(f_i) = \left(\prod_{i=1}^{d} |t_i|\right)^{-1} \delta^d(F).$$

Example 5.5.13. Let $F(x) = g(x)f(x)$, where $f, g : M \to \mathbb{R}$ are analytic functions, such that 0 is a regular value of f and g does not vanish on $f^{-1}(0)$. Then arguing as above, we get

$$\begin{aligned}\delta(f)\mathrm{d}f = \operatorname{sgn}(g(x))\delta(f\cdot g)\mathrm{d}(f\cdot g) &= |g|\delta(g\cdot f)\mathrm{d}f + \operatorname{sgn}(g)f\delta(g\cdot f)\mathrm{d}g \\ &= |g|\delta(g\cdot f)\mathrm{d}f.\end{aligned}$$

Hence

$$\delta(g\cdot f) = |g|^{-1}\delta(f).$$

Proposition 5.5.14. *Suppose $F : M \to \mathbb{R}^D$ is as above and let $j : N = \bigcap_{i=1}^{d} V(f_i) \to M$ be the inclusion. If $u \in Db_M(M)$ is a distribution on M such that j is non-characteristic for $SS(u)$, then the product $\delta^D(F)\cdot u$ is well-defined and given on $\phi \in \Gamma_c(M, C_M^{\infty,\vee})$ by*

$$\langle \delta^D(F)\cdot u, \phi\rangle = \langle u|_N, \phi/\mathrm{d}F\rangle.$$

Proof. That j is non-characteristic for $SS(u)$ means that $SS(u) \cap iT_N^*M$ is contained in the zero section. Prop. 5.4.1 and Prop. 5.5.9 show that the distributions $u|_Y$ and $\delta^D(F)\cdot u$ are well-defined. If $u \in C_M^\infty(M)$, then

$$\langle \delta^D(F)\cdot u, \phi\rangle = \langle \delta^D(F), u\cdot\phi\rangle = \int_N (u\phi)/\mathrm{d}F = \langle u|_N, \phi/\mathrm{d}F\rangle.$$

The general case now follows by continuity of the restriction map by Prop. 5.4.2. □

Example 5.5.15. Let $G = (g_1, \dots, g_r) : M \to \mathbb{R}^r$ be as in Exam. 5.5.8 and suppose additionally that $D \cup N$ is a simple normal crossing divisor, i.e. the intersections $D_I \cap N$ are transverse. Then the product

$$\delta^d(F)u(\lambda) = \prod_{i=1}^{d}\delta(f_i)\prod_{j=1}^{r}|g_j|^{\lambda_j}$$

is well-defined. If $\mathrm{Re}(\lambda_j) > -1$ for all $j \in \{1, \dots, r\}$ then $u(\lambda)$ is locally integral and the above distribution is given by

$$\langle \delta^d(F)u(\lambda), \phi\rangle = \int_{N\setminus D\cap N}\prod_{j=1}^{r}|g_j|^{\lambda_j}\phi/\mathrm{d}F,$$

where the right hand side is a absolutely convergent integral.

Proposition 5.5.16. *Let $F = (f_1, \dots, f_m) : X \to \mathbb{C}^m$ be a holomorphic function which is real-valued on M. Set*

$$\mathcal{L}_F = \{(x, i\xi) \in iT^*M \mid \xi = \sum_{i=1}^{m}\alpha_i\mathrm{d}f_i(x), \alpha_i \geq 0, \alpha_i f_i = 0, i = 1, \dots, m\},$$

and let S_F *be the subset of points* $x \in M$*, such that the system of equations*

$$\sum_{i=1}^{m} \alpha_i \mathrm{d} f_i(x) = 0,$$
$$\alpha_i f_i(x) = 0,$$

has a solution $\alpha \in \mathbb{R}_+^m \backslash \{0\}$*. Then the distribution*

$$u_F(x, \lambda) = \prod_{i=1}^{m} (f_i(x) + i0)^{-\lambda_i}$$

is well-defined on $M \backslash S_F$*, holomorphic in* $\lambda \in \mathbb{C}^m$ *and its singular support satisfies*

$$SS(u(x,\lambda)|_{M\backslash S_F}) \subseteq \mathcal{L}_F \cap \pi^{-1}(M \backslash S_F).$$

Proof. The singular support of the external product $\prod_{i=1}^m (x_i + i0)^{-\lambda_i}$ satisfies

$$SS\left(\prod_{i=1}^{m} (x_i + i0)^{-\lambda_i}\right) \subseteq \{(x, i\sum_{i=1}^{m} \alpha_i \mathrm{d} x_i) \in iT^*\mathbb{R}^m \mid \alpha_i \in \mathbb{R}_{\geq 0}, \alpha_i x_i = 0\}.$$

The pullback

$$u_F(x, \lambda) := F^* \left(\prod_{i=1}^{m} (x_i + i0)^{-\lambda_i}\right)$$

is well-defined if there is no $(x, i\xi) \in SS(\prod_{i=1}^m (x_i + i0)^{-\lambda_i})$, such that $\xi \neq 0$ and $F^*\xi = 0$. The above description of the singular support shows that this is satisfied if $x \notin S_F$. The description of $SS(u_F(x,\lambda))$ is then immediate from Prop. 5.4.1. □

5.6 Distributions on toric varieties

We will also need a way to describe distributional densities on the real locus of a toric variety in terms of homogeneous coordinates. Let X_Σ be a smooth, n-dimensional, toric variety without torus factors, associated to the fan Σ and torus T_N. Recall that we have a toric map

$$\pi_\Sigma : \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma \to X_\Sigma,$$

exhibiting X_Σ as the geometric quotient $X_\Sigma = \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma // G_\Sigma$, where

$$G_\Sigma = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_\rho t_\rho^{\langle m, u_\rho \rangle} = 1 \text{ for all } m \in M\}.$$

The corresponding real locus can then be expressed as

$$X_\Sigma(\mathbb{R}) = \mathbb{R}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R}) // G_\Sigma(\mathbb{R}).$$

An element $t \in G_\Sigma(\mathbb{R})$, acts on distributions $\mu \in Db_{\mathbb{R}^{\Sigma(1)}}(U)$ by duality:

$$\langle t \cdot \mu, \phi \rangle = \langle \mu, (t^{-1})^* \phi \rangle, \qquad \text{for } \phi \in \Gamma_c(U, C^{\infty,n}_{\mathbb{R}^{\Sigma(1)}}).$$

Suppose $\chi \in \hat{G}_\Sigma \cong \mathrm{Cl}(X_\Sigma)$ is a character and $U \subseteq \mathbb{R}^{\Sigma(1)}$ is $G_\Sigma(\mathbb{R})$-invariant. Let us call a distribution $\mu \in Db_{\mathbb{R}^{\Sigma(1)}}(U)$ $|\chi|$-*homogeneous*, if $t \cdot \mu = |\chi(t)|\mu$ for all $t \in G_\Sigma(\mathbb{R})$. Note that for $t = (t_\rho) \in G$ and $\chi = [\sum \chi_\rho e_\rho]$, we have

$$|\chi(t)| = \prod_{\rho \in \Sigma(1)} |t_\rho|^{\chi_\rho}.$$

Let $\sigma \in \Sigma(n)$ and $U_\sigma \subseteq X_\Sigma$ the corresponding affine open subset. Recall from Prop. 3.4.6 that we have isomorphisms $U_\sigma \cong Y_\sigma$ and $\pi_\Sigma^{-1}(U_\sigma) \cong G_\Sigma \times Y_\sigma$, where

$$Y_\sigma = \{x_\rho \in \mathbb{C}^{\Sigma(1)} \mid x_\rho = 1 \text{ for } \rho \notin \sigma(1)\} \subseteq \mathbb{C}^{\Sigma(1)}.$$

Theorem 5.6.1. *Let $\chi_0 = [-\sum_{\rho \in \Sigma(1)} e_\rho] \in \mathrm{Cl}(X_\Sigma)$. Suppose $U \subseteq \mathbb{R}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R})$ is a $G_\Sigma(\mathbb{R})$-invariant open subset and $\mu \in Db_{\mathbb{R}^{\Sigma(1)}}(U)$ is a χ_0-homogeneous distribution. Then there is a unique distributional density $\tilde{\mu}|\Omega|_{X_\Sigma} \in Db^\vee_{X_\Sigma(\mathbb{R})}(\pi_\Sigma(U))$, such that in the local coordinates of each maximal cone $\sigma \in \Sigma(n)$:*

$$\tilde{\mu}|\Omega|_{X_\Sigma}\big|_{\pi_\Sigma(U) \cap U_\sigma(\mathbb{R})} = \mu\big|_{Y_\sigma(\mathbb{R})} \left| \bigwedge_{\rho \in \sigma(1)} \mathrm{d}x_\rho \right|.$$

Under the identification $Y_\sigma(\mathbb{R}) \cong U_\sigma(\mathbb{R})$, its singular support is given by

$$SS(\tilde{\mu}|\Omega|_{X_\Sigma}) = \bigcup_{\sigma \in \Sigma(n)} SS(\mu\big|_{\pi_\Sigma^{-1}(U) \cap Y_\sigma(\mathbb{R})}).$$

Lemma 5.6.2. *The restriction $\mu\big|_{Y_\sigma(\mathbb{R})}$ is well-defined for $\sigma \in \Sigma(n)$.*

Proof. Define L_Σ by the exact sequence

$$0 \longrightarrow L_\Sigma \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow N \longrightarrow 0.$$

For $l \in L_\Sigma$, set $\theta_l = \sum_{\rho \in \Sigma(1)} l_\rho x_\rho \frac{\partial}{\partial x_\rho}$. Differentiating the χ_0-homogeneity condition at $t_0 = 1 \in G_\Sigma(\mathbb{R})$ gives

$$(\theta_l - \langle \chi_0, l \rangle)\mu = 0, \text{ for all } l \in L_\Sigma.$$

Hence μ is a solution to the system

$$M_{L_\Sigma} = \mathcal{D}_{\mathbb{C}^{\Sigma(1)}} / \mathcal{D}_{\mathbb{C}^{\Sigma(1)}} \langle \theta_l - \langle \chi_0, l \rangle \mid l \in L_\Sigma \rangle.$$

It follows from Prop. 5.2.12, that

$$SS(\mu) \subseteq \{(x, i \sum_\rho \xi_\rho \mathrm{d}x_\rho) \in iT^*(\mathbb{R}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R})) \mid \sum_\rho l_\rho x_\rho \xi_\rho = 0 \text{ for } l \in L_\Sigma\}.$$

Let $(x, i\xi) \in SS(\mu)$ with $\xi = \sum_\rho \xi_\rho \mathrm{d}x_\rho$. In order to show that the restriction $\mu\big|_{Y_\sigma}$ is well-defined, we must show that $\xi_\rho = 0$ for $\rho \in \sigma(1)$ and $x_\rho = 1$ for $\rho \notin \sigma(1)$ implies $\xi = 0$. Let $\rho \in \Sigma(1)\backslash\sigma(1)$. Since the ray generators u_{ρ_i} for $\rho_i \in \sigma(1)$ furnish a $\mathbb{Z}$-basis of N, we can find a relation

$$l_\rho u_\rho + \sum_{\rho_i \in \sigma(1)} l_{\rho_i} u_{\rho_i} = 0,$$

with $l_\rho \neq 0$. Set $l = l_\rho e_\rho + \sum_{\rho_i \in \sigma(1)} l_{\rho_i} e_{\rho_i}$. Then $l \in L_\Sigma$ by definition of L_Σ and thus

$$l_\rho x_\rho \xi_\rho + \sum_{\rho_i \in \sigma(1)} l_{\rho_i} x_{\rho_i} \xi_{\rho_i} = l_\rho \xi_\rho = 0.$$

Thus $\xi = 0$ and the restriction is well-defined. □

Proof of Thm. 5.6.1. The uniqueness is clear since the open sets $U_\sigma(\mathbb{R})$ cover $X_\Sigma(\mathbb{R})$. The previous lemma shows that the local expressions are well-defined, so we are left to show that they glue in the right way. Let $\sigma, \tilde{\sigma} \in \Sigma(n)$ be two maximal cones with corresponding local coordinates $(x_\rho)_{\rho \in \sigma(1)}$ and $(\tilde{x}_{\tilde{\rho}})_{\tilde{\rho} \in \sigma(1)}$. We can lift these coordinates to $\mathbb{R}^{\Sigma(1)}$ by setting $x_\rho = 1$ for $\rho \notin \sigma(1)$ and similarly for $\tilde{x}_{\tilde{\rho}}$.

Since the Haar measure on $T_N(\mathbb{R})$ is invariant under torus automorphisms, we have the equality

$$\left| \frac{\mathrm{d}x_{\rho_1}}{x_{\rho_1}} \wedge \ldots \wedge \frac{\mathrm{d}x_{\rho_n}}{x_{\rho_n}} \right| = \left| \frac{\mathrm{d}\tilde{x}_{\tilde{\rho}_1}}{\tilde{x}_{\tilde{\rho}_1}} \wedge \ldots \wedge \frac{\mathrm{d}\tilde{x}_{\tilde{\rho}_n}}{\tilde{x}_{\tilde{\rho}_n}} \right|$$

and therefore

$$|\mathrm{d}x_{\rho_1} \wedge \ldots \wedge \mathrm{d}x_{\rho_n}| = \frac{\prod_{\rho \in \sigma(1)} |x_\rho|}{\prod_{\tilde{\rho} \in \tilde{\sigma}(1)} |\tilde{x}_{\tilde{\rho}}|} |\mathrm{d}\tilde{x}_{\tilde{\rho}_1} \wedge \ldots \wedge \mathrm{d}\tilde{x}_{\tilde{\rho}_n}|$$

Now let $\tilde{\mu}_\sigma = \mu\big|_{Y_\sigma}(x_\rho)$ and $\tilde{\mu}_{\tilde{\sigma}} = \mu\big|_{Y_{\tilde{\sigma}}}(\tilde{x}_{\tilde{\rho}})$. Define $t = (t_\rho) \in (\mathbb{R}\backslash\{0\})^{\Sigma(1)}$ by $t_\rho = \frac{x_\rho}{\tilde{x}_\rho}$, where we use the above convention to regard x_ρ and $\tilde{x}_{\tilde{\rho}}$ as coordinates on $\mathbb{R}^{\Sigma(1)}$. If these coordinates correspond to the point $x \in U_\sigma(\mathbb{R}) \cap U_{\tilde{\sigma}}(\mathbb{R})$, then $t \in G_\Sigma(\mathbb{R})$ and we have

$$\mu_\sigma(x_\rho) = \mu(t \cdot \tilde{x}_{\tilde{\rho}}) = \mu_{\tilde{\sigma}}(\tilde{x}_{\tilde{\rho}}) |\chi_0(t)|,$$

where

$$\begin{aligned} |\chi_0(t)| &= \prod_{\rho \in \Sigma(1)} |t_\rho|^{\langle \chi_0, e_\rho \rangle} = \prod_{\rho \in \Sigma(1)} \left| \frac{x_\rho}{\tilde{x}_\rho} \right|^{-1} \\ &= \frac{\prod_{\tilde{\rho} \in \tilde{\sigma}(1)} |\tilde{x}_{\tilde{\rho}}|}{\prod_{\rho \in \sigma(1)} |x_\rho|}. \end{aligned}$$

This means that

$$\mu_\sigma(x_\rho) \, |\mathrm{d}x_{\rho_1} \wedge \ldots \wedge \mathrm{d}x_{\rho_n}| = \mu_{\tilde{\sigma}}(\tilde{x}_{\tilde{\rho}}) \, |\mathrm{d}\tilde{x}_{\tilde{\rho}_1} \wedge \ldots \wedge \mathrm{d}\tilde{x}_{\tilde{\rho}_n}|$$

and the local expressions glue to a well-defined distributional density $\tilde{\mu}|\Omega|_{X_\Sigma}$ on $\pi_\Sigma(U)$. □

Example 5.6.3. Let $F = (f_1, \ldots, f_d) : U \subseteq \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma \to \mathbb{C}^m$ be an algebraic map, which is real-valued on $U(\mathbb{R})$. Suppose f_i is χ_i-homogeneous for the action of G_Σ, i.e. $f_i(t \cdot x) = \chi_i(t) f_i(x) = \prod_\rho t_\rho^{\chi_{i\rho}} f_i(x)$ for $t \in G_\Sigma$, and assume the zero set $Y = \bigcap_{i=1}^d V(f_i)$ is smooth of codimension d. Then the distribution $\delta^d(F) \in Db_{\mathbb{R}^{\Sigma(1)}}(U(\mathbb{R}))$ is well-defined.

Note that

$$t^* \delta(f_i(x)) = \delta(f_i(t \cdot x)) = \delta(\chi_i(t) f_i(x)) = |\chi_i(t^{-1})| \delta(f_i(x)),$$

by Example 5.5.12. Hence $\delta^d(F)$ is a $|\chi_F|$-homogeneous distribution, where $\chi_F = -\sum_i \chi_i$.

If $u \in Db_{\mathbb{R}^{\Sigma(1)}}(U(\mathbb{R}))$ is another distribution, such that the inclusion $Y(\mathbb{R}) \cap U(\mathbb{R}) \hookrightarrow U(\mathbb{R})$ is non-characteristic for $SS(u)$, and u is $|\chi|$-homogeneous with

$$\chi = \sum_{i=1}^d \chi_i - \left[\sum_{\rho \in \Sigma(1)} e_\rho \right],$$

then the distribution $\mu = \delta^d(F(x)) u(x)$ is well-defined and $|\chi_0|$-homogeneous. Hence $\mu_0 |\Omega|_{X_\Sigma} = \delta^d(F(x)) u(x) |\Omega|_{X_\Sigma}$ is a distributional density on $\pi_\Sigma(U)$ with support in $V = \pi_\Sigma(Y)$.

Remark 5.6.4. Note that on the open subset $U \cap \pi_\Sigma^{-1}(U_\sigma) \cong U \cap (G_\Sigma \times Y_\sigma)$, we have

$$f_i(x) = f_i(t \cdot \tilde{x}) = \chi_i(t) f(\tilde{x}).$$

Since $\chi_i(t)$ is non-vanishing, we get the isomorphism

$$Y \cap p^{-1}(U_\sigma) = \bigcap_{i=1}^d V(f\big|_{U_\sigma}(x)) = G_\Sigma \times \bigcap_{i=1}^d V(f|_{Y_\sigma}(\tilde{x})) = G_\Sigma \times (V \cap U_\sigma).$$

It follows that $V = \pi_\Sigma(Y)$ is smooth of codimension d if and only if Y is.

Proposition 5.6.5. *Suppose X_Σ is compact and Z is another real analytic manifold. If $\mu \in Db_{\mathbb{R}^{\Sigma(G)} \times Z}(\mathbb{R}^{\Sigma(1)} \times Z)$ satisfies the homogeneity condition in Thm. 5.6.1, then the singular support of the integral $I(\mu) = \int_{X_\Sigma(\mathbb{R})} \tilde{\mu} |\Omega|_{X_\Sigma}$ satisfies*

$$SS(I(\mu)) \subseteq \{(z, i\zeta) \in iT^*Z \mid (x, z, 0, i\zeta) \in SS(\mu) \textit{ for some } x \in \mathbb{R}^{\Sigma(1)} \backslash Z_\sigma(\mathbb{R})\}$$

Proof. By Cor. 5.4.10, we have the containment

$$SS(I(\mu)) \subseteq \{(z, i\zeta) \in iT^*Y \mid (x, z, 0, i\zeta) \in SS(\tilde{\mu}|\Omega|_{X_\Sigma}) \text{ for some } \tilde{x} \in X_\Sigma(\mathbb{R})\}.$$

Suppose $\tilde{x}$ above is contained in the affine open $U_\sigma \cong Y_\sigma$ and consider the lift $x = (1, \tilde{x}) \in G_\Sigma(\mathbb{R}) \times Y_\sigma(\mathbb{R}) = p_\Sigma^{-1}(U_\sigma)(\mathbb{R}) \subseteq \mathbb{R}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R})$. The proof of Lemma 5.6.2 then shows that

$$(\tilde{x}, z, 0, i\zeta) \in SS(\tilde{\mu}|\Omega|_{X_\Sigma})|_{\pi^{-1}(U_\sigma \times Z)} = SS(\mu|_{Y_\sigma \times Z})$$

if and only if $(x, z, 0, i\zeta) \in SS(\mu)$. ∎

Let us also note the following variant of Thm. 5.6.1.

Proposition 5.6.6. *Suppose* $\mu \in Db_{\mathbb{R}^{\Sigma(1)}}(\mathbb{R}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R}))$ *is a distribution with support contained in* $\mathbb{R}_{\geq 0}^{\Sigma(1)} \backslash Z_\Sigma(\mathbb{R}_{\geq 0})$ *and such that*

$$t^* \mu = \chi_0(t) \mu,$$

for all $t \in G_\Sigma(\mathbb{R}_{\geq 0})$. *Then there is a unique distributional density* $\tilde{\mu}|\Omega|_{X_\Sigma}$ *on* $X_\Sigma(\mathbb{R})$, *with support in* $X_\Sigma(\mathbb{R}_{\geq 0})$ *and such that in the local coordinates of each maximal cone* $\sigma \in \Sigma(n)$*:*

$$\tilde{\mu}|\Omega|_{X_\Sigma}\big|_{\pi_\Sigma(U) \cap U_\sigma(\mathbb{R})} = \mu\big|_{Y_\sigma(\mathbb{R})} \left| \bigwedge_{\rho \in \sigma(1)} \mathrm{d}x_\rho \right|.$$

Proof. The proof of Lem. 5.6.2 applies without change, so the above local expression is well-defined. To show that they agree on the overlap, it is enough to consider a point $x \in U_\sigma(\mathbb{R}_{\geq 0}) \cap U_{\tilde{\sigma}}(\mathbb{R}_{\geq 0})$, since μ vanishes outside the real-positive locus. Then defining $t \in (\mathbb{R} \backslash \{0\})^{\Sigma(1)}$ by $t_\rho = \frac{x_\rho}{x_{\tilde{\rho}}}$ gives a point $t \in G_\Sigma(\mathbb{R}_{\geq 0})$ and we can argue exactly as in the proof of Thm. 5.6.1. $\square$

6 Graphs and amplitudes

In this chapter, we consider analytically regularized Feynman amplitudes in momentum space. We construct an explicit compactification of the integration domain, and express them in terms of suitable boundary values. This allows us to derive precise results on their meromorphic structure and singular support.

6.1 Feynman propagator

In $D = 1 + (D-1)$ dimensions we consider the Minkoswki inner product

$$p \cdot q = p_0 q_0 - \sum_{i=1}^{D-1} p_i q_i.$$

For $p = q$, we will also write $p^2 = p \cdot p$. The scalar propagator is defined as

$$\Delta(p,m) = \frac{1}{p^2 - m^2 + i0} = b_{\{\mathrm{Im}(p^2)>0\}} \left(\frac{1}{p^2 - m^2} \right)$$

where $m \geq 0$, is the (possibly vanishing) mass of the particle. For the purposes of regularization, let us also define the analytically regularized propagator

$$\Delta(p,m,\lambda) = \frac{1}{p^2 - m^2 + i0} = b_{\{\mathrm{Im}(p^2)>0\}} \left(\frac{1}{p^2 - m^2} \right)^{\lambda}$$

for $\lambda \in \mathbb{C}$.

The following Proposition is then a special case of Prop. 5.5.16

Proposition 6.1.1. *If $m > 0$, then $\Delta(p,m,\lambda)$ is a well-defined distribution on $\mathbb{R}^D$. Its singular support satisfies*

$$SS(\Delta(p,m,\lambda)) \subseteq \{(p, i\alpha p \cdot \mathrm{d}p) \in iT^*\mathbb{R}^D \mid \alpha \in \mathbb{R}_{\geq 0}, \alpha(p^2 - m^2) = 0\}.$$

In the massless case $m = 0$, $\Delta(p,0,\lambda)$ is still well-defined on $\mathbb{R}^D \backslash \{0\}$ and its singular support satisfies

$$SS(\Delta(p,0,\lambda)) \subseteq \{(p, i\alpha p \cdot \mathrm{d}p) \in iT^*(\mathbb{R}^D \backslash \{0\}) \mid \alpha \in \mathbb{R}_{\geq 0}, \alpha p^2 = 0\}.$$

We will need to extend $\Delta(p,m,\lambda)$ to suitable compactification. Let us first consider the massive case $m > 0$. Let $P^D(\mathbb{R})$ be the projective compactification of $\mathbb{R}^D$. Recall from Example 3.4.7, that we have the quotient description $P^D = \mathbb{C}^{D+1}\backslash\{0\}//\mathbb{C}^*$. We denote its homogeneous coordinates by $(u, P_0, \ldots, P_{D-1})$.

By homogenization, it is natural to consider

$$\Delta(u, P, m, \lambda) = |u|^{2\lambda-D-1}(P^2 - u^2m^2 + i0)^{-\lambda}$$

Proposition 6.1.2. *Applying the construction of Thm. 5.6.1 gives a well-defined meromorphic density $\Delta(u, p, m, \lambda)|\Omega|_{P^D} \in \Gamma(P^D(\mathbb{R}), Db^\vee_{P^D(\mathbb{R})})$ with simple poles at*

$$\lambda = \frac{D}{2} - \frac{1}{2}\mathbb{N}.$$

On $\mathbb{R}^D \subseteq P^D(\mathbb{R})$, it agrees with $\Delta(p, m, \lambda)\mathrm{d}^D p$.

Proof. It follows as above, that the boundary value $(P^2 - u^2m^2 + i0)^{-\lambda}$ is well-defined on $\mathbb{R}^{D+1}\backslash\{0\}$ and its singular support is contained in the set

$$\{(u, P, i(-2m^2)\alpha u\mathrm{d}u, i\alpha P \cdot \mathrm{d}P) \in iT^*\mathbb{R}^{D+1} \mid \alpha \in \mathbb{R}_{\geq 0}, \alpha(P^2 - u^2m^2) = 0\}.$$

We also have

$$SS(|u|^{2\lambda-D}) \subseteq \{(u, P, i\alpha\mathrm{d}u, 0) \in iT^*\mathbb{R}^{D+1} \mid \alpha \in \mathbb{R}_{\geq 0}, \alpha u = 0\}.$$

Then $(u, P, i\gamma, i\Xi) \in SS((P^2 - u^2m^2 + i0)^{-\lambda}) \cap SS(|u|^{2\lambda-D})^a$ implies that $\Xi = 0$ and

$$\gamma = -2m^2\alpha_1 u\mathrm{d}u = \alpha_2\mathrm{d}u$$

for some $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{\geq 0}$ satisfying $\alpha_2 u = 0$ and $\alpha_1(P^2 - m^2u^2) = 0$. Thus either $u = 0$ or $\alpha_2 = 0$. Both cases imply $\gamma = 0$ and the product $|u|^{2\lambda-D}(P^2 - u^2m^2 + i0)^{-\lambda}$ is well-defined.

Now let $t \in \mathbb{R}\backslash\{0\}$. Since

$$\mathrm{Im}(P^2 - u^2m^2) > 0 \Leftrightarrow \mathrm{Im}((tP)^2 - u^2t^2m^2) > 0,$$

we have

$$\begin{aligned} t^*((P^2 - u^2m^2 + i0)^{-\lambda}) &= b_{\{\mathrm{Im}((tP)^2 - u^2t^2m^2)>0\}}(t^2P^2 - u^2t^2m^2)^{-\lambda} \\ &= b_{\{\mathrm{Im}((P)^2 - u^2m^2)>0\}}(t^2P^2 - u^2t^2m^2)^{-\lambda} \\ &= |t|^{-2\lambda}(P^2 - u^2m^2 + i0)^{-\lambda}. \end{aligned}$$

Then Δ scales as

$$t^*\Delta(u, P, m, \lambda) = |t|^{-D-1}\Delta(u, P, m, \lambda).$$

It follows from Thm. 5.6.1 that $\Delta|\Omega|_{P^D}$ descends to a meromorphic distributional density on P^D. $\square$

In the massless case, we must also account for the cone point $p = 0$. Let $B^D = Bl_0 P^D$ be the blow-up of P^D in $p = 0$. Recall from Example 3.4.8, that

$$B^D \cong \mathbb{C}^{D+2} \backslash Z_{B^D} // (\mathbb{C}^*)^2,$$

with homogeneous coordinates $(s, u, P_0, \dots, P_{D-1})$ and

$$Z_{B^D} = \bigcap_{i=0}^{D-1} V(ux_i) \cap V(sx_i).$$

Then $E_s = V(s)$ defines the exceptional divisor and $E_u = V(u)$ the hyperplane at infinity. We have $\mathrm{Cl}(B^D) \cong \mathbb{Z}^2$ and the variables have scaling degrees

$$\deg(s) = (0, -1), \deg(u) = (1, 0), \deg(P_i) = (1, 1).$$

We then let

$$\Delta_0(s, u, P, \lambda) = |u|^{2\lambda - D - 1} |s|^{D-1-2\lambda} (P^2 + i0)^{-\lambda}$$

Proposition 6.1.3. *$\Delta_0 |\Omega|_{B^D}$ defines a meromorphic density on $B^D(\mathbb{R})$, with simple poles on*

$$\lambda \in \frac{D}{2} + \frac{1}{2}\mathbb{Z}.$$

The restriction to $\mathbb{R}^D \backslash \{0\} \cong B^D(\mathbb{R}) \backslash (E_u(\mathbb{R}) \cup E_s(\mathbb{R}))$ is well-defined for arbitrarily λ and agrees with $\Delta(p, 0, \lambda) \mathrm{d}^D p$ there.

Proof. Note that the cone point $P = 0$ lies in Z_{B^D}, so the factor $(P^2 + i0)^{-\lambda}$ is well-defined on $B^D(\mathbb{R}) \backslash Z_{B^D}(\mathbb{R})$. The external product

$$\Delta_0(s, u, P, \lambda) = |u|^{2\lambda - D - 1} |s|^{D-1-2\lambda} (P^2 + i0)^{-\lambda}$$

is thus well-defined there. The factor $|u|^{2\lambda - D - 1}$ produces simple poles for $\lambda \in \frac{D}{2} - 1/2\mathbb{N}$ and $|s|^{D-1-2\lambda}$ produces simple poles for $\lambda \in \frac{D}{2} + 1/2\mathbb{N}$. Note that for $\lambda = \frac{D}{2}$ both factors are singular, but this still produces only a simple pole, since u and s can not vanish simultaneously on $B^D(\mathbb{R}) \backslash Z_{B^D}(\mathbb{R})$.

An element $t = (t_1, t_2) \in (\mathbb{R} \backslash \{0\})^2$ acts as

$$t \cdot P_i = \frac{t_2}{t_1} P_i, \quad t \cdot u = t_2 u, \quad t \cdot s = t_1 s.$$

As above we find

$$t^*(P^2 + i0)^{-\lambda} = \left| \frac{t_2}{t_1} \right|^{-2\lambda} (P^2 + i0)^{-\lambda}$$

and thus

$$t^* \Delta_0(s, u, P) = |t_1|^{D-1} |t_2|^{-D-1} \Delta_0(s, u, P),$$

i.e. $\Delta_0 |\Omega|_{B^D}$ has the right scaling properties and descends to a meromorphic distributional density on $B^D(\mathbb{R})$. $\square$

Remark 6.1.4. Let $f : B^D \backslash E_u(\mathbb{R}) \to \mathbb{R}^D$ be the blow-up map. Then the pushforward

$$f_!(\Delta_0 |\Omega|_{B^D}) \in \Gamma(\mathbb{R}^D, Db^\vee_{\mathbb{R}^D})$$

is a meromorphic extension of $(p^2 + i0)^{-\lambda} \mathrm{d}^D p \in \Gamma(\mathbb{R}^D \backslash \{0\}, Db^\vee_{\mathbb{R}^D})$ with poles for $\lambda \in \frac{D}{2} + 1/2\mathbb{N}$.

6.2 Feynman graphs

We will consider a *graph* G to consist of a triple

$$G = (E_G, V_G, \partial)$$

of finite sets of edges E_G and vertices V_G, together with a map

$$\partial = \partial_G : E_G \to \operatorname{Sym}^2 V_G = V_G \times V_G / \mathbb{Z}_2,$$

mapping an edge to its endpoints. This definition allows multiple edges and loops, but our graphs will not have external half-edges. We will often write $i \in G$ for $i \in E_G$ and denote by $|G| = |E_G|$ the number of edges of G.

An edge $i \in G$ is called a *selfloop* if $\partial(e) = (v, v)$. A subgraph $\gamma \subseteq G$ is given by subsets $E_\gamma \subseteq E_G, V_\gamma \subseteq V_G$, such that $\partial(E_\gamma) \subseteq \operatorname{Sym}^2 V_\gamma$.

Every graph has an obvious geometric realization as a one-dimensional CW-complex, so we can speak about topological notions like connectedness. In particular, we denote by $h^0(G)$ and $h^1(G)$ the first and second Betti numbers of (the geometric realization of) G. For a connected graph G, we call a connected subgraph $T \subseteq G$ a *spanning tree* if $V_T = V_G$ and $h^1(T) = 0$. Note that these are precisely the maximal simply-connected subgraphs of G.

A spanning 2-tree is a simply-connected subgraph $F \subseteq G$, with $V_F = V_G$ and exactly two connected components $F = T_1 \cup T_2$. Every spanning 2-tree is obtained from a spanning tree by deleting an edge.

Every subset $I \subseteq E_G$ gives the *edge subgraph* $\gamma \subseteq G$, where $E_\gamma = I$ and V_γ consists of all vertices incident to an edge in I. We call two edge subgraphs $\gamma_1, \gamma_2 \subseteq G$ *edge-disjoint* if their edge sets E_{γ_i} are disjoint. We almost exclusively deal with edge subgraphs, so we will often identify an edge subgraph with its set of edges. Notable exceptions are spanning 2-trees, where it is important to allow isolated vertices.

If $\gamma \subseteq G$ is an edge subgraph with connected components $\gamma_1, \ldots, \gamma_k$, we define the the quotient graph by contracting edge connected component $\gamma_i \subseteq G$ to a point. More precisely, we set

$$E_{G/\gamma} = E_G \backslash E_\gamma, \quad V_{G/\gamma} = V_G \cup V_\gamma \cup \{v_{\gamma_i} \mid i = 1, \ldots, k\}$$

and $\partial_{G/\gamma}(j) = \partial_G(j)$ if j does not end in a vertex of γ, and $\partial_{G/\gamma}(j) = \{v_{\gamma_a}, v_{\gamma_b}\}$ if j connects the two subgraphs γ_a and γ_b (permitting the case $\gamma_a = \gamma_b$).

Similarly, we define the deletion $G\backslash\gamma$ of G by γ by deleting the edges of γ, i.e.

$$E_{G\backslash\gamma} = E_G \backslash E_\gamma, \quad V_{G\backslash\gamma} = V_G, \quad \partial_{G\backslash\gamma} = \partial_G|_{E_G\backslash E_\gamma}$$

A *Feynman graph* is a graph G together with distinguished sets of external vertices $V_G^{ext} \subseteq V$ and massive edges $E_G^M \subseteq E_G$. We will assume that $|V^{ext}| \geq 2$.

To every external vertex $v \in V_G^{ext}$ we associate an inflowing external momentum $p_v \in \mathbb{R}^D$ and to every massive edge $i \in E_G^m$, a mass $m_i \in (0, \infty)$. Let us also define $E_G^0 = E_G^m \backslash E_G$, the set of massless edges and $V_G^{int} = V_G \backslash V_G^{ext}$, the set of internal vertices. It will be convenient to set $p_v = 0$ for $v \in V_G^{int}$ and $m_e = 0$ for $i \in E_G^0$.

A Feynman graph is called *massive* if $E_G^M = E_G$, i.e. every edge carries a non-vanishing mass. Similarly, we call G *massless* if $E_G^0 = E_G$. A subgraph $\gamma \subseteq G$ is called massless if it consists solely of massless edges. We denote by $G^0 \subseteq G$ the maximal massless subgraph G, i.e. the (possibly empty) edge-subgraph consisting of all massless edges.

If $\gamma \subseteq G$ is an edge subgraph, then the masses of G/γ and $G\backslash\gamma$ are inherited from G through the inclusion $E_{G/\gamma} = E_{G\backslash\gamma} \subseteq E_G$. The external momenta of $G\backslash\gamma$ are the same as those of G. The quotient graph inherits the external momenta of G through the obvious surjection $V_G \to V_{G/\gamma}$.

Let us introduce the following notation: For a finite set A and another set M, we set $A(M) := M^A$. If M is an abelian group, then $A(M) = \mathbb{Z}A \otimes_{\mathbb{Z}} M$. We apply this in particular for $A = E_G$ and $A = V_G$, the set of edges resp. vertices of a graph G.

Choosing an orientation for each edge gives G the structure of a one-dimensional cell complex. We can then refine the edge boundary map $\partial : E_G \to \mathrm{Sym}^2(V_G)$ to a map $(\partial_+, \partial_-) : E_G \to V_G^2$, where $\partial_+(e)$ (resp. $\partial_-(e)$) is the target (resp. source) vertex of the oriented edge e.

The cellular chain complex gives the exact sequence

$$0 \longrightarrow H_1(G, \mathbb{Z}) \xrightarrow{i} E_G(\mathbb{Z}) \xrightarrow{\partial} V_G(\mathbb{Z}) \longrightarrow H_0(G, \mathbb{Z}) \longrightarrow 0$$

The following then follows by a simple diagram chase.

Proposition 6.2.1. *For a subgraph $\gamma \subseteq G$ we have the following diagram with exact rows and columns.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_1(\gamma, \mathbb{Z}) \arrow[r] \arrow[d] & E_\gamma(\mathbb{Z}) \arrow[r, "\partial_\gamma"] \arrow[d] & V_\gamma(\mathbb{Z}) \arrow[r] \arrow[d] & H_0(\gamma, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(G, \mathbb{Z}) \arrow[r] \arrow[d] & E_G(\mathbb{Z}) \arrow[r, "\partial_G"] \arrow[d] & V_G(\mathbb{Z}) \arrow[r] \arrow[d] & H_0(G, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(G/\gamma, \mathbb{Z}) \arrow[r] \arrow[d] & E_{G/\gamma}(\mathbb{Z}) \arrow[r, "\partial_{G/\gamma}"] \arrow[d] & V_{G/\gamma}(\mathbb{Z}) \arrow[r] \arrow[d] & H_0(G/\gamma, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $\tilde{V}_G^0(\mathbb{Z}) = \ker(V_G(\mathbb{Z}) \to H_0(G, \mathbb{Z}))$ and $V_G^0(\mathbb{Z}) = \tilde{V}_G^0(\mathbb{Z}) \cap V_G^{ext}(\mathbb{Z})$. The space $V_G^0(\mathbb{Z})$ is the subspace of external momenta satisfying momentum conservation

$$\sum_{v \in V_{G_0}^{ext}} p_v = 0$$

in each connected component $G_0 \subseteq G$.

A simple consequence of the above exact sequences is the following behaviour of the first Betti number $h^1(-)$.

Proposition 6.2.2. *Let $\gamma, \eta \subseteq G$ be two subgraphs. The first Betti numbers satisfies*

$$h^1(\gamma) + h^1(G/\gamma) = h^1(G)$$
$$h^1(\gamma \cup \eta) + h^1(\gamma \cap \eta) \geq h^1(\gamma) + h^1(\eta).$$

Proof. The first equation is obtained by taking the ranks in the leftmost exact column above. For two subgraphs $\gamma, \eta \subseteq G$, we have the truncated Mayer-Vietoris sequence:

$$0 \longrightarrow H_1(\gamma \cap \eta, \mathbb{Z}) \longrightarrow H_1(\gamma, \mathbb{Z}) \oplus H_1(\eta, \mathbb{Z}) \longrightarrow H_1(\gamma \cup \eta, \mathbb{Z}) \longrightarrow \tilde{H} \longrightarrow 0$$

where

$$\tilde{H} = H_1(\gamma \cup \eta, \mathbb{Z})/(H_1(\gamma, \mathbb{Z}) \oplus H_1(\eta, \mathbb{Z})).$$

Taking ranks gives

$$h^1(\gamma \cup \eta) + h^1(\gamma \cap \eta) = h^1(\gamma) + h^1(\eta) + \operatorname{rk} \tilde{H} \geq h^1(\gamma) + h^1(\eta).$$

□

Let us recall Kruskal's algorithm, which gives a convenient way to construct spanning trees. Suppose G is a connected graph and $E_G = \{j_1, \ldots, j_{|G|}\}$ an enumeration of its edges. Set $T_0 = \emptyset$ and let $T_k = T_{k-1} \cup \{j_k\}$ if $h^1(T_{k-1} \cup j_k) = 0$ and $T_k = T_{k-1}$ otherwise.

Proposition 6.2.3. *The edge subgraph $T_{|G|} \subseteq G$ is a spanning tree of G.*

Proof. By construction we have $h^1(T_{|G|}) = 0$. Suppose $T_{|G|}$ is not a spanning tree, i.e. it is not a maximal edge-subgraph with the property $h^1(T_{|G|}) = 0$. Then there would be an edge $i = i_k \in E_G$ not contained in $T_{|G|}$ such that $h^1(T_i \cup i_k) = 0$. But $i_k \notin T_{|G|}$ implies $h^1(T_{k-1} \cup i_k) > 0$, a contradiction. □

A spanning tree $T \subseteq G$ induces a basis of $H_1(G, \mathbb{Z})$:

Proposition 6.2.4. *Let $T \subseteq G$ be a spanning tree. There is a natural isomorphism $H_1(G, \mathbb{Z}) \cong E_{G \setminus T}(\mathbb{Z})$. For each $j \in E_G \setminus E_T$, there is unique generator C_j of $H_1(G, Z)$ of the form*

$$C_j = e_j + \sum_{i \in T} C_{ji} e_i.$$

If T is constructed by Kruskal's algorithm through an enumeration $E_G = \{j_1, \ldots, j_{|G|}\}$, then $C_{j_k j_l} = 0$ unless $j_k > j_l$.

Proof. We have the following diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & E_T(\mathbb{Z}) \arrow[r] \arrow[d] & V_T(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(G,\mathbb{Z}) \arrow[r] \arrow[d] & E_G(\mathbb{Z}) \arrow[r] \arrow[d] & V_G(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(G/T,\mathbb{Z}) \arrow[r] \arrow[d] & E_{G\setminus T}(\mathbb{Z}) \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
 & 0 & 0 & & &
\end{tikzcd}$$

In particular we have isomorphisms $H_1(G,\mathbb{Z}) \cong H_1(G/T,\mathbb{Z}) \cong E_{G\setminus T}(\mathbb{Z})$. For every edge $j \in E_{G\setminus T}$ we have $H_1(T \cup e_j, \mathbb{Z}) \cong \mathbb{Z}$ with a generator of the form

$$C_j = e_j + \sum_{i \in T} C_{ji} e_i.$$

Since $H_1(T \cup \{j\}, \mathbb{Z}) \to H_1(G,\mathbb{Z})$ is injective, its image in $H_1(G,\mathbb{Z})$ is again a generator. The isomorphism $E_{G\setminus T}(\mathbb{Z}) \cong H_1(G,\mathbb{Z})$ is then given by $e_j \mapsto C_j$.

Now if $j = j_k$ in the above enumeration, then $H_1(T_{k-1} \cup j_k) \cong \mathbb{Z}$, which means the circuit C_{j_k} is supported by the edges in $E_{T_{k-1}} \cup \{j_k\}$, i.e $C_{j_k j_l} = 0$ for $l > k$. □

Suppose $\gamma \subseteq G$ is an edge subgraph with connected components $\gamma_1, \dots, \gamma_k$. We call a spanning tree $T \subseteq G$ *adapted to* γ if $T \cap \gamma_i$ is a spanning tree for each connected component $\gamma_i \subseteq \gamma$. This is again equivalent to $T \cap \gamma$ being a maximal subgraph with $h^1(T \cap \gamma) = 0$.

Proposition 6.2.5. *Let $T \subseteq G$ be a spanning tree and $\gamma \subseteq G$ an edge subgraph. Then*

$$\begin{aligned} |\gamma \cap T| &\leq |\gamma| - h^1(\gamma), \\ |T/T \cap \gamma| &\geq |G/\gamma| - h^1(G/\gamma), \end{aligned}$$

with equality if and only if T is adapted to γ.

Proof. The number of edges $|\gamma|$ as well as $h^1(\gamma)$ is additive over the connected components. The first inequality then follows from the previous proposition, with equality if and only if $|\gamma_i \cap T|$ is a maximal tree for all i, which means that T is adapted to γ. For the contracted graph, we get

$$\begin{aligned} |T/T \cap \gamma| &= |T| - |T \cap \gamma| \leq |G| - h^1(G) - (|\gamma| - h^1(\gamma)) \\ &= |G/\gamma| - (h^1(G) - h^1(\gamma)) = |G/\gamma| - h^1(G/\gamma), \end{aligned}$$

with equality if and only if $|T \cap \gamma| = |\gamma| - h^1(\gamma)$, i.e. iff T is adapted to γ. □

Remark 6.2.6. Note that $T/T\cap\gamma$ contains all vertices of G/γ and for every spanning tree $\tilde{T}\subseteq T/\gamma\cap T$ of G/γ, we have $|\tilde{T}| = |T/T\cap\gamma| = |G/\gamma| - h^1(G/\gamma)$ if and only if T is adapted to γ.

Corollary 6.2.7. *Suppose $\gamma\subseteq\Gamma\subseteq G$ is a flag of subgraphs and $T\subseteq G$ a spanning tree which is adapted to γ and Γ. Then the cycles*

$$\tilde{C}_j = e_j + \sum_{i\in T\cap(\Gamma\backslash\gamma)} C_{ji}e_i, \quad j\in\Gamma\backslash(T\cap\Gamma\cup\gamma)$$

are a basis for $H_1(\Gamma/\gamma,\mathbb{Z})$.

Proof. They are clearly linear independent elements of $E_{\Gamma/\gamma}(\mathbb{Z})$ and are obtained from the cycles C_j of Prop. 6.2.4 by applying the projection $E_\Gamma(Z)\to E_{\Gamma/\gamma}(\mathbb{Z})$. Since the boundary map ∂ commutes with contraction, they are cycles in $H_1(\Gamma/\gamma,\mathbb{Z})$. By the above proposition, we also have

$$|\Gamma\backslash(T\cap\Gamma\cup\gamma)| = |\Gamma/\gamma| - |T\cap\Gamma/T\cap\gamma| = h^1(\Gamma/\gamma),$$

which means that the above cycles generate $H_1(\Gamma/\gamma,\mathbb{Z})$. □

For a Feynman graph G, we define the *completion* G^∞ of G by adding a vertex v_∞ and an edge $e_a\in E_{G^\infty}$ for each external vertex $v_a\in V_G^{ext}$, such that e_a joins v_∞ and v_a. More formally, we let $V_{G^\infty} = V_G\cup\{\infty\}$ and $E_{G^\infty} = E_G\cup V_G^{ext}$. The boundary map is given by ∂_G on $E_G\subseteq E_{G^\infty}$ and by $\partial(e_a) = \{v_\infty, v_a\}$ on the new edges. By convention, we regard the edge e_a as oriented away from v_∞.

6.3 Feynman integrals

To a connected Feynman graph G, we associate the formal integral

$$I_G(\lambda,p) = \int_{E_G(\mathbb{R}^D)} \prod_{v\in V_G} \delta^D(p_v+k_v) \prod_{j\in G} (k_j^2 - m_j^2 + i0)^{-\lambda_j} \mathrm{d}^D k_j,$$

where $p\in V_G(\mathbb{R}^D), m\in E_G(\mathbb{R}_{\geq 0})$ and $\lambda\in\mathbb{C}^{E_G}$. Here k_v is the projection of $\partial k\in V_G(\mathbb{R}^D)$ on to the factor corresponding to $v\in V_G$, i.e.

$$k_v = \sum_{i\in G}\epsilon_{iv}k_i,$$

where

$$\epsilon_{iv} = \begin{cases} +1, & \partial_+(i) = v \\ -1, & \partial_-(i) = v \\ 0, & i\notin\partial_+^{-1}(v)\cup\partial_-^{-1}(v). \end{cases}$$

Let us then write $I_G(\lambda,p) = \int_{E_G(\mathbb{R}^D)} J_G(\lambda,p,k)\prod_{i\in G}\mathrm{d}^D k_i$.

Proposition 6.3.1. *The integrand J_G is well-defined as a distribution on $V_G^{ext}(\mathbb{R}^D) \times E_G^m(\mathbb{R}^D) \times E_G^0(\mathbb{R}^D \backslash \{0\})$.*

Let us fix some notation before embarking on the proof. We identify $\mathbb{R}^D$ with its dual through the Minkowski inner product. Then we have canonical isomorphism $T^*\mathbb{R}^D \cong \mathbb{R}^D \times \mathbb{R}^D$. For $p = (p_v) \in V_G(\mathbb{R}^D)$ we write the covector coordinate as $x = (x_v) \in V(\mathbb{R}^D)$, since momentum space is dual to position space. Note that we have the dual exact sequence

$$0 \longleftarrow H^1(G, \mathbb{R}^D) \xleftarrow{i^*} E_G(\mathbb{R}^D) \xleftarrow{\delta} V_G(\mathbb{R}^D) \xleftarrow{\delta} H^0(G, \mathbb{R}^D) \longleftarrow 0.$$

Recall that we will identify $V_G^{ext}(\mathbb{R}^D)$ with a subspace of $V_G(\mathbb{R}^D)$ by setting $p_v = 0$ for $v \notin V_G^{ext}$. We will also need the following lemma.

Lemma 6.3.2. *Let $\alpha \in E_G(\mathbb{R}_{\geq 0}), k \in E_G(\mathbb{R})$ and define $\alpha \cdot k \in E_G(\mathbb{R})$ by $(\alpha \cdot k)_i = \alpha_i k_i$. Suppose there is $x \in V_G(\mathbb{R})$ such that $\delta x = \alpha \cdot k$ and $\langle x, \partial k \rangle = 0$. Then $\alpha \cdot k = 0$.*

Proof. Since ∂ and δ are adjoint to each other, we have

$$0 = \langle x, \partial k \rangle = \langle \delta x, k \rangle = \langle \alpha \cdot k, k \rangle = \sum_{j \in G} \alpha_j k_j^2.$$

For $\alpha_j \geq 0$, this is only possible if $\alpha \cdot k = 0$. $\square$

Proof of Prop. 6.3.1. Section 6.1 shows that the external product $\prod_j (k_j^2 - m_j^2 + i0)^{-\lambda_j}$ is well-defined on $E_G^m(\mathbb{R}^D) \times E_G^0(\mathbb{R}^D \backslash \{0\})$ for fixed $\lambda \in E_G(\mathbb{C})$ and its singular support is contained in the set of points

$$(k, p, i\xi, i\zeta) \in iT^*(E_G^m(\mathbb{R}^D) \times E_G^0(\mathbb{R}^D \backslash \{0\}) \times V_G^{ext}(\mathbb{R}^D))$$

satisfying $\zeta = \sum_j \alpha_j k_j \cdot \mathrm{d}k_j$ for $\alpha \in E_G(\mathbb{R}_{\geq 0})$ with $\alpha_j(k_j^2 - m_j^2) = 0$.

Similarly the product $\prod_v \delta_v^D(p_v + k_v)$ defines the subspace

$$\begin{aligned} Y &= \bigcap_v V(p_v - k_v) = \partial^{-1}(V_G^0(\mathbb{R}^D)) \\ &\cong H_1(G, \mathbb{R}^D) \oplus V_G^0(\mathbb{R}^D), \end{aligned}$$

which is of codimension $D|V_G|$ by Euler's formula. This product is then well-defined by Prop. 5.5.9.

Since the map $x \mapsto \delta x$ is dual to $k \mapsto \partial k$, the singular support $SS(\prod_{v \in V_G} \delta_v^D(p_v + k_v))$ is contained in the set of points

$$(k, p, i\zeta) \in iT^*(E_G(\mathbb{R}^D) \times V_G^{ext}(\mathbb{R}^D)),$$

satisfying $p = -\partial k$ and

$$\zeta = \sum_{j \in G} (\delta x)_j \cdot \mathrm{d}k_j + \sum_{b \in V_G^{ext}} x_b \cdot \mathrm{d}p_b,$$

for some $x \in V_G(\mathbb{R}^D)$.

If additionally, $(k, p, i\zeta) \in SS(\prod_j (k_j^2 - m_j^2)^{-\lambda_i})$, then $x_b = 0$ for $b \in V_G^{ext}$ and thus, for every spacetime component $\beta \in \{0, \dots, D-1\}$, the triple $(\alpha, k^\beta, x^\beta)$ satisfies the assumptions of the above lemma. Hence $\alpha \cdot k = 0$ and thus $\zeta = 0$. It follows that the product $\prod_v \delta_v^D(p_v + k_v) \prod_j (k_j^2 - m_j^2 + i0)^{-\lambda_j}$ is well-defined. $\square$

It will be useful to have an alternative description of the delta-function product $\prod_v \delta^D(k_v + p_v)$. Let G^∞ be the completion of G. Under the bijection $E_{G^\infty} = E_G \cup V_G^{ext}$ we can consider the pair $k = (k, p) \in E_G(\mathbb{R}^D) \times V_G^{ext}(\mathbb{R}^D)$ as momenta $\tilde{k} \in E_{G^\infty}(\mathbb{R}^D)$.

Lemma 6.3.3. *Under the above identifications, we have $\partial_G k + p = 0$ if and only if $\partial_{G^\infty} \tilde{k} = 0$. Similarly, $\tilde{k}|_{H_1(G^\infty, \mathbb{R}^D)} = 0$ if and only if there is $x \in V_G(\mathbb{R}^D)$ with $k = \delta_G x$ and $x_a = p_a$ for $a \in V_G^{ext}$.*

Proof. Considering G as a subgraph of G^∞ gives the commutative diagram

$$\begin{tikzcd}
E_G(\mathbb{R}^D) \arrow[r, "\partial_G"] \arrow[d] & V_G(\mathbb{R}) \arrow[d] \\
E_G \oplus V_G^{ext}(\mathbb{R}^D) \arrow[r, "\partial_{G^\infty}"] & V_{G^\infty}(\mathbb{R}^D) \oplus \mathbb{R}^D\langle v_\infty \rangle
\end{tikzcd}$$

For edges e_a corresponding to external momenta, we have $\partial_{G^\infty} e_a = v_a - v_\infty$. This gives

$$\begin{aligned}
\partial_{G^\infty} \tilde{k} &= \partial_G k + \sum_{a \in V_G^{ext}} \partial_{G^\infty} p_a e_a \\
&= \partial_G k + \sum_{a \in V_G^{ext}} p_a v_a - \sum_{a \in V_G^{ext}} p_a v_\infty.
\end{aligned}$$

Comparing coefficients gives $\partial_{G^\infty} \tilde{k} = 0$ if and only if $\partial_G k + p = 0$ and $\sum_{a \in V_G^{ext}} p_a = 0$. But the last condition already follows from $\partial_G k + p = 0$, since then

$$\sum_{a \in V_G^{ext}} p_a = \sum_{v \in V_G} p_v = -\sum_{v \in V_G} (\partial_G k)_v = -\partial_G^2 k = 0.$$

Now suppose $\tilde{k}$, considered as an element of the dual $(E_{G^\infty}(\mathbb{R}^D))^* \cong E_{G^\infty}(\mathbb{R}^D)$, vanishes on $H_1(G^\infty, \mathbb{R}^D) \subseteq E_{G^\infty}(\mathbb{R}^D)$. Since

$$\ker(E_{G^\infty}(\mathbb{R}^D) \to H^1(G^\infty, \mathbb{R}^D)) \cong V_{G^\infty}(\mathbb{R}^D)/H^0(G^\infty, \mathbb{R}^D),$$

we can find $x \in V_{G^\infty}(\mathbb{R}^D)$ such that $x_{v_\infty} = 0$ and $\tilde{k} = \delta_{G^\infty} x$. Then $\tilde{k}_j = \sum_{v \in V_{G^\infty}} \epsilon_{jv} x_v$ and comparing coefficients shows that this equation is equivalent to

$$\begin{aligned}
p_a &= x_a, \quad a \in V_G^{ext} \\
k_j &= \sum_{v \in V_G} \epsilon_{jv} x_v = \delta_G x.
\end{aligned}$$

Conversely if $x \in V_G(\mathbb{R}^D)$ satisfies $\delta_G x = k$ and $x_a = p_a$, then setting $x_{v_\infty} = 0$ gives an element $x \in V_{G^\infty}(\mathbb{R}^D)$ with $\tilde{k} = \delta_{G^\infty} x$. Then

$$\tilde{k}|_{H_1(G^\infty,\mathbb{R}^D)} = \delta x|_{H_1(G^\infty,\mathbb{R}^D)} = 0.$$

□

Let $T^\infty \subseteq G^\infty$ be a spanning tree. Recall from Prop. 6.2.4, that every edge $j \in G^\infty \backslash T^\infty$ gives the circuit

$$C_j = e_j + \sum_{i \in T^\infty} C_{ji} e_i$$

and the collection of these circuits furnishes a basis of $H_1(G^\infty, \mathbb{Z})$.

Proposition 6.3.4. *Suppose $T^\infty \subseteq G^\infty$ is a spanning tree. Let $T_1 = T^\infty \cap G$, $T_2 = T^\infty \cap (G^\infty \backslash G)$ and V_2 be the subset of V_G^{ext} corresponding to $E_{T_2} \subseteq E_{G^\infty \backslash G}$.*

1. *The equation $\partial_G k + p = 0$ is equivalent to the system of equations*

$$k_i = \sum_{j \in G \backslash T_1} C_{ji} k_j + \sum_{a \in V_G^{ext} \backslash V_2} C_{e_a i} p_a, \quad i \in T_1$$
$$p_b = \sum_{j \in G \backslash T_1} C_{j e_b} k_j + \sum_{a \in V_G^{ext} \backslash V_2} C_{e_a e_b} p_a, \quad b \in V_2$$

2. *The condition $k = \delta_G x$ for $x \in V_G(\mathbb{R}^D)$ with $x_a = p_a, a \in V_G^{ext}$, is equivalent to the system of equations*

$$k_j = -\sum_{i \in T_1} C_{ji} k_i - \sum_{b \in V_2} C_{j e_b} p_b, \quad j \in G \backslash T_1$$
$$p_a = -\sum_{i \in T_1} C_{e_a i} k_i - \sum_{b \in V_2} C_{e_a e_b} p_b, \quad a \in V_G^{ext} \backslash V_2$$

Proof. Let $\tilde{k} \in E_{G^\infty}(\mathbb{R}^D)$ correspond to $(k,p) \in E_G(\mathbb{R}^D) \times V_G^{ext}(\mathbb{R}^D)$.

1. The equation $\partial_G k + p = 0$ is equivalent to $\tilde{k} \in H_1(G^\infty, \mathbb{R}^D)$. Thus $\tilde{k}$ can be uniquely expressed as

$$\begin{aligned} \tilde{k} &= \sum_{j \in G^\infty \backslash T^\infty} \tilde{k}_j C_j \\ &= \sum_{j \in G^\infty \backslash T^\infty} \left(\tilde{k}_j e_j + \sum_{i \in T^\infty} C_{ji} \tilde{k}_j e_i \right) \end{aligned}$$

   Comparing coefficients gives $\partial_G k + p = 0$ if and only if

$$k_i = \sum_{j \in G \backslash T_1} C_{ji} k_j + \sum_{a \in V_G^{ext} \backslash V_2} C_{e_a i} p_a, \quad i \in T_1$$
$$p_b = \sum_{j \in G \backslash T_1} C_{j e_b} k_j + \sum_{a \in V_G^{ext} \backslash V_2} C_{e_a e_b} p_a, \quad b \in V_2.$$

2. The condition $k = \delta_G x$ with x as above is equivalent to $\delta_{G^\infty}\tilde{k} = 0$. Since the C_j for $j \in T^\infty$ furnish a basis of $H_1(G^\infty, \mathbb{Z})$, this is equivalent to

$$0 = \langle \tilde{k}, C_j \rangle = \tilde{k}_j + \sum_{i \in T^\infty} C_{ji}\tilde{k}_i,$$

for all $j \in G^\infty \backslash T^\infty$. Expressing $\tilde{k}$ and T^∞ in terms of (k, p) and $T_1 \cup V_2$ then gives

$$\begin{aligned} k_j &= -\sum_{i \in T_1} C_{ji}k_i - \sum_{b \in V_2} C_{je_b}p_b, \quad j \in G\backslash T_1 \\ p_a &= -\sum_{i \in T_1} C_{e_a i}k_i - \sum_{b \in V_2} C_{e_a e_b}p_b, \quad a \in V_G^{ext}\backslash V_2. \end{aligned}$$

□

The integral

$$I_G(\lambda, p) = \int_{E_G^m(\mathbb{R}^D) \times E_G^0(\mathbb{R}^D\backslash\{0\})} J_G(\lambda, p, k) \prod_{i \in G} \mathrm{d}^D k_e$$

is still ill-defined, since the support of J_G is not proper along the projection

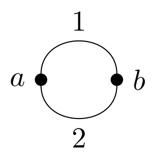
$$E_G^m(\mathbb{R}^D) \times E_G^0(\mathbb{R}^D\backslash\{0\}) \times V_G^{ext}(\mathbb{R}^D) \to V_G^{ext}(\mathbb{R}^D).$$

A natural compactification of the fibre would be the product

$$\overline{P_G} = \prod_{i \in E_G^m} P^D \times \prod_{i \in E_G^0} B^D.$$

We have seen that the external product of propagators $\prod_{i \in G} \Delta(k_i, m_i, \lambda_i)$ naturally extends to a meromorphic distribution on $\overline{P_G}(\mathbb{R})$. Unfortunately, the closure $\overline{V}$ of $\bigcap_v V(p_v + k_v)$ in $\overline{P}_G(\mathbb{R}^D) \times V_G(\mathbb{R}^D)$ is not smooth in general and it is not clear how to extend the delta-function $\prod_v \delta^v(k_v + p_v)$.

Example 6.3.5. Consider the bubble graph[1]



and let us assume that the masses m_i are non-vanishing and the edges e_1 and e_2 are oriented from v_a to v_b. The equation $\partial k + p = 0$ is then equivalent to the system

$$\begin{aligned} f_1(k, p) &= p_a + p_b = 0 \\ f_2(k, p) &= k_1 + k_2 + p_b = 0. \end{aligned}$$

[1]The graphs in this thesis were drawn with TikZ-Feynman [Ell17].

In homogeneous coordinates, where $k_i = K_i/u_i$, the first equation is unchanged, while the second becomes

$$f(c,K,p) = \frac{1}{u_1u_2}(u_2K_1 + u_1K_2 + u_1u_2p_a) = 0.$$

In a coordinate system around the point $u_1 = u_2 = 0$, $\overline{V}$ is then defined by the equations

$$\begin{aligned} p_a + p_b &= 0 \\ u_2K_1 + u_1K_2 + u_1u_2p_a &= 0, \end{aligned}$$

where K_1 and K_2 have one coordinate normalized to 1. The differential of

$$\tilde{f}_2(u,K,p) = u_2K_1 + u_1K_2 + u_1u_2p_a$$

at a point where $u_1 = u_2 = 0$ is then

$$\mathrm{d}\tilde{f}_2(u,K,p)\big|_{u_1=u_2=0} = K_1\mathrm{d}u_2 + K_2\mathrm{d}u_1.$$

Thus the rank of $\mathrm{d}\tilde{f}_2$ is strictly smaller then D for $D > 2$ and $\overline{V}$ is singular at points where $u_1 = u_2 = 0$.

Luckily, an explicit desingularization of $\overline{V}$ has been given by Sato et. al. in [SMJO76]. Moreover, it fits naturally in to the theory we have developed in Section 3.8. We will construct a compactification P_G as the toric variety obtained by an iterated blowup of

$$P_G = \prod_{i\in G} P^D.$$

We have a natural poset isomorphism

$$(\Sigma_{P_G}, \preceq) \cong \prod_{i\in G}(2^{\mathcal{D}}\backslash\mathcal{D}, \subseteq) := (\mathcal{L}_G, \preceq),$$

where $\mathcal{D} = \{u, 0, \ldots, D-1\}$ corresponds to the set of spacetime coordinates plus an additional homogeneous coordinate $u \in \mathcal{D}$ for the hyperplane at infinity. Every element $L \in \mathcal{L}_G$ can then be written as $L = \prod_i L_i$, where $L_i \subsetneq \mathcal{D}$ and $L \preceq \tilde{L}$ if and only if $L_i \subseteq \tilde{L}_i$ for all $i \in G$.

Let $\gamma \subseteq G$ and $\eta \subseteq G$ be two edge-subgraph, such that η is massless, i.e. $\eta \subseteq G^0$. We define corresponding elements $F_\gamma, F_\eta \in \mathcal{L}_G$ by

$$F_\gamma = \prod_{i\in\gamma}\{u\} \times \prod_{i\notin\gamma}\emptyset$$

and

$$F_\eta = \prod_{i\in\eta}\{0,\ldots,D-1\} \times \prod_{i\notin\eta}\emptyset.$$

For $j \in G$ and $\beta \in \mathcal{D}$, we also set

$$F_{j,\beta} = \prod_{\substack{\tilde{j} \in G \\ \tilde{j} \neq j}} \emptyset \times \prod_{\tilde{j}=j} \{\beta\}.$$

For $\beta = u$ it will also be convenient to write $F_{j,u} = F_j$. The corresponding ray generators $u_F \in E_G(\mathbb{Z}^D) \cong \mathbb{Z}^{E_G} \otimes \mathbb{Z}^D$ are given by

$$\begin{aligned}
u_{F_\gamma} &= \sum_{j \in \gamma} \sum_{\beta=0}^{D-1} e_j \otimes (-e_\beta), \quad \gamma \subseteq G \\
u_{F_\eta} &= \sum_{j \in \eta} \sum_{\beta=0}^{D-1} e_j \otimes e_\beta, \quad \eta \subseteq G^0 \\
u_{F_{j,u}} &= \sum_{\beta=0}^{D-1} e_j \otimes (-e_\beta) \\
u_{F_{j,\beta}} &= e_j \otimes e_\beta, \quad \beta \neq u.
\end{aligned}$$

Note that the $F_{j,\beta}$ are precisely the atoms of $\mathcal{L}_G$. Then consider the set

$$\mathcal{G} = \{F_\gamma \mid \gamma \subseteq G\} \cup \{F_\eta \mid \eta \subseteq G^0\} \cup \{F_{j,\beta} \mid j \in G, \beta \in \mathcal{D}\}.$$

Lemma 6.3.6. *$\mathcal{G} \subseteq \mathcal{L}_G \backslash \hat{0}$ is a building set in $\mathcal{L}_G$.*

Proof. Suppose $L \in \mathcal{L}_G \backslash \hat{0}$, such that $L = \prod_i L_i$. Define edge subgraphs γ_L, η_L by the corresponding edge subsets

$$\begin{aligned}
E_{\gamma_L} &= \{i \in G \mid u \in L_i\} \\
E_{\eta_L} &= \{i \in G^0 \mid \{0, \dots, D-1\} \subseteq L_i\}.
\end{aligned}$$

It follows from $L_i \subsetneq \mathcal{D}$ for all $i \in G$, that γ and η are edge-disjoint and we have

$$\max \mathcal{G}^{\preceq L} = \{F_{\gamma_L}, F_{\eta_L}\} \cup \{F_{i,\beta} \mid \alpha \in L_i, i \in \gamma_F, \alpha \neq u\} \cup \{F_{i,\beta} \mid \alpha \in L_i, i \notin \gamma_F \cup \eta_F\}.$$

Then

$$\begin{aligned}
\prod_{F \in \max \mathcal{G}^{\preceq L}} [\hat{0}, F] &\cong [\hat{0}, F_{\gamma_L}] \times \prod_{i \in \gamma_L} \prod_{\alpha \in \{0,\dots,D-1\} \cap L_i} [\emptyset, \{\alpha\}] \times [\hat{0}, F_{\eta_L}] \times \prod_{i \notin \gamma_L \cup \eta_L} \prod_{\alpha \in L_i} [\emptyset, \{\alpha\}] \\
&\cong \prod_{i \in \gamma_L} [\emptyset, L_i] \times \prod_{i \in \eta_L} [\emptyset, L_i] \times \prod_{i \notin \gamma_L \cup \eta_L} [\emptyset, L_i] \\
&\cong \prod_{i \in G} [\emptyset, L_i] \cong [\hat{0}, L]
\end{aligned}$$

and $\mathcal{G}$ is a building set. $\square$

Let $P = P_{\mathcal{G}}$ be the iterated blow-up of P_G given by the building set $\mathcal{G}$ as in Section 3.8. From Prop. 3.8.8, we have the homogeneous coordinate description

$$P_{\mathcal{G}} = (\mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}) // (\mathbb{C}^*)^{E_G} \times (\mathbb{C}^*)^{\mathcal{G}^*},$$

where $\mathcal{G}^* \subseteq \mathcal{G}$ consists of the non-atomic elements:

$$\mathcal{G}^* = \{F_\gamma \mid \gamma \subseteq G, |E_\gamma| \geq 2\} \sqcup \{F_\eta \mid \eta \subseteq G^0\} =: \mathcal{G}_1^* \sqcup \mathcal{G}_2^*.$$

Let

$$p_{\mathcal{G}} : (\mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}) \to P_{\mathcal{G}}$$

be the quotient map. We denote the homogeneous coordinate corresponding to F_γ by u_γ and the one corresponding to F_η by s_η. If γ consists of the single edge i then we write $u_i = u_\gamma$. For $i \in G$ write $K_i = (K_{i,0}, \dots, K_{i,D-1})$ for the homogeneous coordinate corresponding to the momentum flowing through the ith edge. The inhomogeneous coordinates $(k_i)_{i \in G}$ of $E_G(\mathbb{R}^D)$ can then be expressed as

$$k_i = K_i \prod_{i \in \gamma \subseteq G} u_\gamma^{-1} \prod_{i \in \eta \subseteq G^0} s_\eta,$$

where the products extend over all subgraphs of G (resp. G^0) which contain the edge i. Let us define

$$U_i := \prod_{i \in \gamma \subseteq G} u_\gamma, \quad S_i := \prod_{i \in \eta \subseteq G^0} s_\eta,$$

so that the above coordinate expression takes the form $k_i = \frac{S_i}{U_i} K_i$.

Lemma 6.3.7. *An element $t \in (\mathbb{C}^*)^{E_G} \times (\mathbb{C}^*)^{\mathcal{G}^*}$ acts on these coordinates as*

$$t \cdot K_i = t_i \prod_{\substack{\gamma \in \mathcal{G}_1^* \\ i \in \gamma}} t_\gamma \prod_{\substack{\eta \in \mathcal{G}_2^* \\ i \in \eta}} t_\eta^{-1} K_i, \quad t \cdot s_i = t_i s_i \quad t \cdot u_\gamma = t_\gamma u_\gamma, \quad t \cdot s_\eta = t_\eta s_\eta.$$

In $\mathrm{Cl}(P_{\mathcal{G}})$*, we also have the equality*

$$\chi_0 := -\left[\sum_{F \in \mathcal{G}} e_F\right] = \left[\sum_{\gamma \subseteq G} (-D|\gamma| - 1) e_{F_\gamma} + \sum_{\eta \subseteq G^0} (D|\eta| - 1) e_{F_\eta}\right]$$

Proof. The group $(\mathbb{C}^*)^{E_G} \times (\mathbb{C}^*)^{\mathcal{G}^*}$ acts diagonally on $\mathbb{C}^{\mathcal{G}}$ under the isomorphism

$$\begin{aligned}(\mathbb{C}^*)^{E_G} \times (\mathbb{C}^*)^{\mathcal{G}^*} &\cong \mathrm{Hom}(\mathrm{Cl}(P_{\mathcal{G}}), \mathbb{C}^*) \\ &= \{t \in (\mathbb{C}^*)^{\mathcal{G}} \mid \prod_{F \in \mathcal{G}} t_F^{\langle m, u_F \rangle} = 1, m \in E_G(\mathbb{Z}^D)\}\end{aligned}$$

For $m = e^i \otimes e^\beta \in E_G(\mathbb{Z}^D) \cong \mathbb{Z}^{E_G} \otimes \mathbb{Z}^D$ we get

$$t_{F_{i,\beta}} \left(\prod_{i \in \gamma \subseteq G} t_{F_\gamma} \right)^{-1} \left(\prod_{i \in \eta \subseteq G^0} t_{F_\eta} \right) = 1$$

and thus

$$t_{F_{i,\beta}} = \left(\prod_{i \in \gamma \subseteq G} t_{F_\gamma} \right) \left(\prod_{i \in \eta \subseteq G^0} t_{F_\eta} \right)^{-1},$$

from which the above action on coordinates follows.

Under the isomorphism $\mathrm{Cl}(P_{\mathcal{G}}) \cong \mathbb{Z}^{\mathcal{G}} / \mathbb{Z}^{E_G} \otimes \mathbb{Z}^D$ we have

$$0 = \left[\sum_{F \in \mathcal{G}} \langle e^i \otimes e^\beta, u_F \rangle e_F \right] = \left[e_{F_{i,\beta}} - \sum_{i \in \gamma \subseteq G} e_{F_\gamma} + \sum_{i \in \eta \subseteq G^0} e_{F_\eta} \right]$$

and the above formula for χ_0 follows as well. $\square$

Lemma 6.3.8. *The maximal cones of $\Sigma_{\mathcal{G}}$ are in bijective correspondence with triples*

$$\mathcal{N} = (\mathcal{I}, \mathcal{J}, \beta),$$

where

- $\mathcal{I} = \{\gamma_1 \subsetneq \gamma_2 \subsetneq \ldots \subsetneq \gamma_r\}$ *is a flag of edge subgraphs* $\gamma_i \subseteq G$.
- $\mathcal{J} = \{\eta_1 \subsetneq \eta_2 \subsetneq \ldots \subsetneq \eta_t\}$ *is a flag of edge subgraphs* $\eta_i \subseteq G^0$.
- β *is a map* $\beta : E_{\gamma \cup \eta} \to \{0, \ldots, D-1\}$.

These data are subject to the conditions:

1. γ_r *and* η_t *are edge-disjoint.*
2. $\eta_t = G^0 \backslash (G^0 \cap \gamma_r)$.
3. $|\mathcal{I}| = |\gamma_r|$ *and* $|\mathcal{J}| = |\eta_t|$.

Proof. From Thm. 3.8.6 we know that the maximal cones of $\Sigma_{\mathcal{G}}$ are in bijective correspondence with nested sets $\tilde{\mathcal{N}} \subseteq \mathcal{G}$ such that $|\tilde{\mathcal{N}}| = \dim P_{\mathcal{G}} = D|E_G|$. For such a nested set $\tilde{\mathcal{N}}$, let $\tilde{\mathcal{I}}_0 = \tilde{\mathcal{N}} \cap \mathcal{G}_1^*$ and $\tilde{\mathcal{J}} = \mathcal{G}_2^*$. If $F_{\gamma_1}, F_{\gamma_2} \in \tilde{\mathcal{I}}_0$, then $F_{\gamma_1} \vee F_{\gamma_2} = F_{\gamma_1 \cup \gamma_2} \in \mathcal{G}$. Since $\tilde{\mathcal{N}}$ is a nested set, every two elements $F_{\gamma_1}, F_{\gamma_2}$ must then be pairwise comparable. Hence $\tilde{\mathcal{I}}_0 = \{F_{\gamma_1}, \ldots, F_{\gamma_r}\}$ corresponds to a flag of subgraphs $\mathcal{I}_0$ as above. Similarly, for $\tilde{F}_{\eta_1}, \tilde{F}_{\eta_2} \in \tilde{\mathcal{J}}$, we have $\tilde{F}_{\eta_1} \vee \tilde{F}_{\eta_2} = \tilde{F}_{\eta_1 \cup \eta_2}$ and $\tilde{\mathcal{J}}$ corresponds to a flag $\mathcal{J}$ of subgraphs $\eta_j \subseteq G^0$.

For $F_\gamma \in \tilde{\mathcal{I}}_0$ and $\tilde{F}_\eta \in \tilde{\mathcal{J}}$ the upper bound $F_\gamma \vee \tilde{F}_\eta = L$ must exist in $\mathcal{L}_G$. If $j \in \gamma \cap \eta$ would be a shared edge, then $L_j = \mathcal{D}$ which is impossible by definition of $\mathcal{L}_G$. Hence all pairs $(\gamma, \eta) \in \mathcal{I}_0 \times \mathcal{J}$ must be edge-disjoint.

Now let $\mathcal{I}_1 = \{i \in G \mid F_{i,u} \in \tilde{\mathcal{N}}\}$. It follows again from the nestedness condition, that $\mathcal{I}_1$ can consist of at most a single edge and that $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ is a flag of subgraphs which are pairwise disjoint from elements $\eta \in \mathcal{J}$. Let γ_r and η_t be the maximal elements of $\mathcal{I}$ and $\mathcal{J}$. Suppose

$$\mathcal{B} = \{(i, \alpha) \in E_G \times \{0, \dots, D-1\} \mid F_{i,\beta} \in \tilde{\mathcal{N}}\}$$

is the remaining set of atoms contained in $\tilde{\mathcal{N}}$ and let $\mathcal{B}^c = E_G \times \{0, \dots, D-1\} \backslash \mathcal{B}$ be its complement. For each edge $i \in \gamma_r$, the atoms $\{F_{i,\beta}, \beta \in \{0, \dots, D-1\}\}$ can not all be contained in $\tilde{\mathcal{N}}$, since the subset $\{F_{i,\beta}, F_{\gamma_r}, \beta \in \{0, \dots, D-1\}\}$ does not have an upper bound in $\mathcal{L}_G$. Then there must be $\beta(i) \in \{0, \dots, D-1\}$ such that $F_{i,\beta(i)} \in \mathcal{B}^c$.

Similarly, if $i \in G^0$, then $F_{i,0} \vee \ldots \vee F_{i,D-1} = \tilde{F}_i \in \mathcal{G}$ so there must be $\beta(i) \in \{0, \dots, D-1\}$, such that $F_{i,\beta(i)} \in \mathcal{B}^c$. But $|\tilde{\mathcal{N}}| = D|E_G| = |\mathcal{B}| + |\mathcal{B}^c|$ implies that

$$|\mathcal{I}| + |\mathcal{J}| = |\mathcal{B}^c|,$$

and we also have

$$|\gamma_r| + |\eta_t| \geq |\mathcal{I}| + |\mathcal{J}| = |\mathcal{B}^c| \geq |\gamma_r \cup G^0| = |\gamma_r| + |\eta_t| + |G^0 \backslash (\gamma_r \cup \eta_t)|.$$

This implies that $G^0 \subseteq \gamma_r \cup \eta_t$ and the map

$$\beta : E_{\mathcal{G}^0 \cup \gamma_r} = E_{\eta_t \cup \gamma_r} \to \{0, \dots, D-1\}$$

is uniquely determined. We also obtain $|\mathcal{I}| = |\gamma_r|$ and $|\mathcal{J}| = |\eta_t|$.

Conversely, if $(\mathcal{I}, \mathcal{J}, \beta)$ are as above, then it is easy to check that

$$\tilde{\mathcal{N}} = \{F_\gamma \mid \gamma \in \mathcal{I}\} \cup \{F_\eta \mid \eta \in \mathcal{J}\} \cup \{F_{i,\beta} \mid i \notin \gamma_r \cup \eta_t \text{ or } \beta \neq \beta(i)\}$$

is a nested set with $|\tilde{\mathcal{N}}| = D|E_G|$. □

We denote by $\mathcal{U}_\mathcal{N} \subseteq P_\mathcal{G}$ the affine open corresponding to $\mathcal{N}$ and by $\tilde{\mathcal{U}}_\mathcal{N} \subseteq \mathbb{C}^\mathcal{G} \backslash Z_\mathcal{G}$ its preimage under $p_\mathcal{G}$. With the above notation, we get

$$\tilde{\mathcal{U}}_\mathcal{N} = \mathbb{C}^\mathcal{G} \backslash \left(\bigcup_{\gamma \notin \mathcal{I}} V(u_\gamma) \cup \bigcup_{\eta \notin \mathcal{J}} V(s_\eta) \cup \bigcup_{j \in \gamma \cup \eta} V(K_{j\beta(j)}) \right)$$

The inhomogeneous coordinates on $\mathcal{U}_\mathcal{N}$ are obtained by setting

$$\begin{aligned} u_\gamma &= 1, \quad \gamma \notin \mathcal{I}, \\ s_\eta &= 1, \quad \eta \notin \mathcal{J}, \\ K_{i,\beta(i)} &= 1, \quad i \in \gamma_r \cup \eta_t. \end{aligned}$$

On $\mathbb{C}^\mathcal{G}$, we can consider $k_i = \frac{S_i}{U_i} K_i$ as a rational function, which is regular on the open torus $(\mathbb{C}^*)^\mathcal{G}$. Then the subvariety

$$Y_\mathcal{G} = V(\partial k + p) = \bigcap_{v \in V_G} V\left(\sum_{i \in G} \epsilon_{iv} \frac{S_i}{U_i} K_i + p_v \right) \cap (\mathbb{C}^*)^\mathcal{G} \times V_G^{ext}(\mathbb{C}^D),$$

is the preimage under $p_{\mathcal{G}} \times \mathrm{id}_{V_G^{ext}(\mathbb{C}^D)}$ of

$$H_{\mathcal{G}} = \bigcap_{v \in V_G} V(k_v + p_v) \cap (E_G((\mathbb{C}^*)^D) \times V_G^{ext}(\mathbb{C}^D).$$

Its closure $\overline{Y}_{\mathcal{G}} \subseteq \mathbb{C}^{\mathcal{G}} \backslash Z_\Sigma \times V_G^{ext}(\mathbb{C}^D)$ is obtained from the above equations by judiciously clearing denominators on each open subset $\tilde{\mathcal{U}}_{\mathcal{N}}$.

For a point $\overline{z} = (\overline{s}, \overline{u}, \overline{K}) \in \mathbb{C}^{\mathcal{G}} \backslash Z_\Sigma$, let

$$\overline{\mathcal{I}} = \{\gamma \subseteq G \mid \overline{u}_\gamma = 0\}, \quad \overline{\mathcal{J}} = \{\eta \subseteq G^0 \mid \overline{s}_\eta = 0\}.$$

Since $z \notin Z_\Sigma$, we can find a nested set $\mathcal{N} = (\mathcal{I}, \mathcal{J}, \beta)$ such that $\overline{z} \in \tilde{\mathcal{U}}_{\mathcal{N}}$, which implies $\overline{\mathcal{I}} \subseteq \mathcal{I}$ and $\overline{\mathcal{J}} \subseteq \mathcal{J}$ are subflags

$$\overline{\mathcal{I}} = \{\overline{\gamma}_1 \subsetneq \ldots \subsetneq \overline{\gamma}_r\}, \quad \overline{\mathcal{J}} = \{\overline{\eta}_1 \subsetneq \ldots \subsetneq \overline{\eta}_t\}$$

Let

$$\overline{U}_i = \prod_{i \in \gamma \in \overline{\mathcal{I}}} u_\gamma, \quad \overline{S}_i = \prod_{i \in \eta \in \overline{\mathcal{J}}} s_\eta$$

and define the rational function $\overline{k}$ by

$$\begin{aligned}\overline{k}_i &= \begin{cases} \overline{U}_i k_i, & i \in \overline{\gamma}_r \\ \overline{S}_i^{-1} k_i, & i \in \overline{\eta}_t \\ k_i, & i \notin \overline{\gamma}_r \cup \overline{\eta}_t. \end{cases} \\ &= \left(\prod_{i \in \eta \notin \overline{\mathcal{I}}} U_\eta^{-1} \right) \left(\prod_{i \in \eta \notin \overline{\mathcal{J}}} S_\eta \right) K_i\end{aligned}$$

Note that $\overline{k}$ is regular in a neighbourhood $\mathcal{U}_{\overline{z}}$ of $\overline{z}$ and $\overline{k}_i = 0$ differs from K_i by a monomial which is non-vanishing on $\mathcal{U}_{\overline{z}}$.

Proposition 6.3.9. *The closure $\overline{Y}_{\mathcal{G}} \subseteq (\mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}) \times V_G^{ext}(\mathbb{C}^D)$ of $Y_{\mathcal{G}}$ and its image*

$$\overline{H}_{\mathcal{G}} = p_{\mathcal{G}}(\overline{Y}_{\mathcal{G}}) \in P_{\mathcal{G}}(\mathbb{C}) \times V_G^{ext}(\mathbb{C}^D)$$

are smooth subvarieties of codimension $D|V_G|$.

Our proof is adapted from [SMJO76].

Proof. Let $\overline{z} \in \mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}, \mathcal{U}_{\overline{z}}, \overline{\mathcal{I}}, \overline{\mathcal{J}}$ and $\overline{k}$ as above. Let G^∞ be the completion of G and choose an enumeration of $E_{G^\infty} = \{e_1, \ldots, e_N\}$, such that

$$\begin{aligned} E_{\overline{\gamma}_r} &= \{e_1, \ldots, e_{n_1}\}, \\ V^{ext} \cong E_{G^\infty \backslash G} &= \{e_{n_1+1}, \ldots, e_{n_2}\}, \\ E_{G \backslash (\overline{\gamma}_r \cup \overline{\eta}_t)} &= \{e_{n_2+1}, \ldots, e_{n_3}\}, \\ E_{\overline{\eta}_t} &= \{e_{n_3+1}, \ldots, e_N\}. \end{aligned}$$

We also require that the induced ordering of E_{G^∞} is compatible with $\overline{\mathcal{I}}$ and $\overline{\mathcal{J}}^{op}$ in the sense that

$$\begin{aligned} E_{\overline{\gamma}_i} &= \{e_1, \dots, e_{n_i}\} \\ E_{\overline{\nu}_j} &= \{e_{N-n_j+1}, \dots, e_N\}. \end{aligned}$$

By Prop. 6.2.3, we can construct a spanning tree $T^\infty \subseteq G^\infty$ by iteratively selecting edges in the order specified above. The corresponding equations of Prop. 6.3.4 are then

$$\begin{aligned} k_i &= \sum_{j \in \overline{\eta}_t \setminus (\overline{\eta}_t \cap T_1)} C_{ji} k_j, \quad i \in T_1 \cap \overline{\eta}_t, \\ k_i &= \sum_{j \in G \setminus (\overline{\gamma}_r \cup T_1)} C_{ji} k_j, \quad i \in T_1 \setminus T_1 \cap (\overline{\gamma}_r \cup \overline{\eta}_t), \\ p_b &= \sum_{j \in G \setminus (\overline{\gamma}_r \cup T_1)} C_{je_b} k_j + \sum_{a \in V_G^{ext} \setminus V_2} C_{e_a e_b} p_a, \quad b \in V_2, \\ k_i &= \sum_{j \in G \setminus T_1} C_{ji} k_j + \sum_{a \in V_G^{ext} \setminus V_2} C_{e_a i} p_a, \quad i \in T_1 \cap \overline{\gamma}_r. \end{aligned}$$

where we have used that $C_{ji} = 0$ unless $j > i$ in this ordering. In terms of $\overline{k}$, the above system is equivalent to

$$\begin{aligned} \overline{k}_i &= \sum_{j \in \overline{\eta}_t \setminus (\overline{\eta}_t \cap T_1)} C_{ji} \frac{\overline{S}_j}{\overline{S}_i} \overline{k}_j, \quad i \in T_1 \cap \overline{\eta}_t, \\ \overline{k}_i &= \sum_{j \in \overline{\eta}_t \setminus (\overline{\eta}_t \cap T_1)} C_{ji} \overline{S}_j \overline{k}_j + \sum_{j \notin \gamma_r \cup \overline{\eta}_t \cup T_1} C_{ji} \overline{k}_j, \quad i \in T_1 \setminus T_1 \cap (\overline{\gamma}_r \cup \overline{\eta}_t), \\ p_b &= \sum_{j \in \overline{\eta}_t \setminus (\overline{\eta}_t \cap T_1)} C_{je_b} \overline{S}_j \overline{k}_j + \sum_{j \notin \overline{\gamma}_r \cup \overline{\eta}_t \cup T_1} C_{je_b} \overline{k}_j + \sum_{a \in V_G^{ext} \setminus V_2} C_{e_a e_b} p_a, \quad b \in V_2, \\ \overline{k}_i &= \overline{U}_i \left(\sum_{j \in \overline{\eta}_t \setminus (\overline{\eta}_t \cap T_1)} C_{ji} \overline{S}_j \overline{k}_j + \sum_{j \notin \overline{\gamma}_r \cup \overline{\eta}_t \cup T_1} C_{ji} \overline{k}_j + \sum_{a \in V_G^{ext} \setminus V_2} C_{e_a i} p_a \right) \\ &\quad + \sum_{j \in \overline{\gamma}_r \setminus (\overline{\gamma}_r \cap T_1)} C_{ji} \frac{\overline{U}_i}{\overline{U}_j} \overline{k}_j \quad i \in T_1 \cap \overline{\gamma}_r. \end{aligned}$$

We claim that the rational factors appearing on the RHS are well-defined on $\mathcal{U}_{\overline{z}}$. Let us consider the factor $C_{ji} \frac{\overline{S}_i}{\overline{S}_j}$ appearing in the first equation, where $i, j \in \eta_t$ and $i < j$. Since the enumeration is compatible with $\mathcal{J}^{op}$ we have

$$\overline{S}_{ji} := \frac{\overline{S}_j}{\overline{S}_i} = \prod_{\substack{\eta \in \overline{\mathcal{J}} \\ j \in \eta, i \notin \eta}} s_\eta.$$

which is is well-defined on $\mathcal{U}_{\overline{z}}$. Similarly we get

$$\overline{U}_{ji} := \frac{\overline{U}_i}{\overline{U}_j} = \prod_{\substack{\gamma\in\overline{\mathcal{I}} \\ i\in\gamma, j\notin\gamma}} u_\gamma.$$

This system of $D|V_G|$ equations is manifestly independent, so we have shown that the corresponding subvariety $\overline{Y} \cap \mathcal{U}_{\overline{z}}$ is smooth of codimension $D|V|$. Since the open sets $\mathcal{U}_{\overline{z}}$ cover $\mathbb{C}^{\mathcal{G}} \backslash Z_{\mathcal{G}}$, $\overline{Y}$ must be a smooth subvariety of codimension $D|V_G|$. That the same is true for $\overline{H}_{\mathcal{G}}$ now follows from Remark 5.6.4. $\square$

Remark 6.3.10. Let

$$D_{\mathcal{G}} = \bigcup_{\gamma\subseteq G} V(u_\gamma) \cup \bigcup_{\eta\subseteq G^0} V(s_\eta).$$

The explicit local equation for $\overline{Y}_{\mathcal{G}}$ above show that the subvariety $D_{\mathcal{G}} \cup \overline{Y}_{\mathcal{G}}$

$$D_{\mathcal{G}} = \overline{Y}_{\mathcal{G}} \cup \bigcup_{\gamma\subseteq G} V(u_\gamma) \cup \bigcup_{\eta\subseteq G^0} V(s_\eta)$$

and its image $p_{\mathcal{G}}(D_{\mathcal{G}} \cup Y_{\mathcal{G}})$ are simple normal crossing divisors.

Suppose $T^\infty \subseteq G^\infty$ is a spanning tree as in the above proof and let

$$\begin{aligned} f_i(s,z,K,p) &= \sum_{j\in G\backslash T_1} C_{ji}k_j + \sum_{a\in V_G^{ext}\backslash V_2} C_{e_a i}p_a, \quad i\in T_1, \\ g_b(s,z,K,p) &= \sum_{j\in G\backslash T_1} C_{je_b}k_j + \sum_{a\in V_G^{ext}\backslash V_2} C_{e_a e_b}p_a, \quad b\in V_2. \end{aligned}$$

The system of equations $\partial k + p = 0$ is related to the system

$$\begin{aligned} f_i &= k_i, \quad i\in T_1, \\ g_b &= p_b, \quad b\in V_2, \end{aligned}$$

by a linear automorphism L_{T^∞} defined over $\mathbb{Z}$. Hence $\det L_{T^\infty} = \pm 1$ and we have the identity

$$\prod_{v\in V_G} \delta^D\left(\sum_{i\in G} \epsilon_{iv}\frac{S_i}{U_i}K_i + p_v\right) = \prod_{i\in T_1}\delta^D(k_i - f_i)\prod_{b\in V_2}\delta^D(p_b - g_b).$$

Using Example 5.5.13, we also have

$$\prod_{i\in T_1}\delta^D(k_i - f_i)\prod_{b\in V_2}\delta^D(p_b - g_b) = \prod_{i\in T_1}\left(\frac{|\overline{U}_i|}{|\overline{S}_i|}\right)\delta^D\left(\overline{k}_i - \frac{\overline{U}_i}{\overline{S}_i}f_i\right)\prod_{b\in V_2}\delta^D(p_b - g_b).$$

It follows from the above proof, that the product of delta functions on the RHS is well-defined on $\mathcal{U}_{\overline{z}}(\mathbb{R})$ and defines the real part of the subvariety $\overline{Y}_{\mathcal{G}} \cap \mathcal{U}_{\overline{z}}$. The product over T_1 has the following more concrete expression.

Lemma 6.3.11. *With the above choice of spanning tree T^∞ and flags $\overline{\mathcal{I}}, \overline{\mathcal{J}}$ we have*

$$\prod_{i \in T_1} |\overline{U}_i| = \prod_{\overline{\gamma} \in \overline{\mathcal{I}}} |u_{\overline{\gamma}}|^{|\overline{\gamma} \cap T_1|} = \prod_{\overline{\gamma} \in \overline{\mathcal{I}}} |u_{\overline{\gamma}}|^{|\overline{\gamma}| - h^1(\overline{\gamma})}$$
$$\prod_{i \in T_1} |\overline{S}_i| = \prod_{\overline{\eta} \in \overline{\mathcal{J}}} |s_{\overline{\eta}}|^{|\overline{\eta} \cap T_1|} = \prod_{\overline{\eta} \in \overline{\mathcal{J}}} |s_{\overline{\eta}}|^{|\overline{\eta}| - \overline{h}^1(\overline{\eta})},$$

where $\overline{h}^1(\overline{\eta}) = h^1(G^\infty/(G^\infty \backslash \overline{\eta}))$.

Proof. The first equation in each line is immediate from the definitions. By construction, T^∞ is adapted to the subgraphs $\overline{\gamma} \subseteq G^\infty$ and $G^\infty \backslash \overline{\eta}$ for $\overline{\gamma} \in \overline{\mathcal{I}}$ and $\overline{\eta} \in \overline{\mathcal{J}}$. From Prop. 6.2.5, we then get

$$|\overline{\gamma} \cap T_1| = |\overline{\gamma}| - h^1(\overline{\gamma}),$$

as well as

$$|\overline{\eta} \cap T_1| = |\overline{\eta}| - h^1(G^\infty/(G^\infty \backslash \overline{\eta})) = |\overline{\eta}| - \overline{h}^1(\overline{\eta}).$$

□

Let us now come back to our original integrand. In the above homogeneous coordinates, the propagator can be expressed

$$\begin{aligned}(k_j - m_j^2 + i0)^{-\lambda_j} &= \left(\frac{S_j}{U_j} K_j - m_j^2 + i0\right)^{-\lambda_j} \\ &= \overline{\Delta}_j(u, K, \lambda) \prod_{j \in \gamma \subseteq G} |u_\gamma|^{2\lambda_j} \prod_{j \in \eta \subseteq G^0} |s_\eta|^{-2\lambda_j}\end{aligned}$$

where the homogenized propagators are defined as

$$\overline{\Delta}_j(u, K, \lambda) = \begin{cases} (K_j^2 - \prod_{e \in \gamma \subseteq G} u_\gamma^2 m_j^2 + i0)^{-\lambda_j}, & j \in E_G^m \\ (K_j^2 + i0)^{-\lambda_j}, \quad j \in E_G^0 \end{cases}$$

We then consider the (putative) distribution

$$\overline{J}_G(\lambda) = \prod_{j \in G} \overline{\Delta}_j(u, K, \lambda) \prod_{v \in V_G} \delta^D(d_v(s, u, K, p)) \prod_{\gamma \subseteq G} |u_\gamma|^{2\lambda_\gamma - D|\gamma| - 1} \prod_{\eta \subseteq G^0} |s_\eta|^{D|\eta| - 2\lambda_\eta - 1}$$

where

$$\begin{aligned}d_v(s, u, K, p) &= \sum_{j \in G} \epsilon_{jv} \prod_{j \in \eta \subseteq G^0} \frac{S_j}{U_j} K_j - p_v, \\ \lambda_\gamma &= \sum_{j \in \gamma} \lambda_j, \\ \lambda_\eta &= \sum_{j \in \eta} \lambda_j.\end{aligned}$$

Theorem 6.3.12. *$\overline{J}_G(\lambda)$ is a well-defined, meromorphic distribution on $(\mathbb{R}^{\mathcal{G}}\backslash Z_{\mathcal{G}}(\mathbb{R}))\times V_G^{ext}(\mathbb{R}^D)$, with simple poles contained in the hypersurfaces*

$$2\lambda_\gamma - Dh^1(\gamma)\in -\mathbb{N},\quad \gamma\subseteq G$$
$$2\lambda_\eta - D\overline{h}^1(\eta)\in \mathbb{N},\quad \eta\subseteq G^0.$$

Moreover it is $|\chi_0|$-homogeneous, where $\chi_0=[-\sum_{F\in\mathcal{G}} e_F]$ and the corresponding meromorphic distributional density $\overline{J}_G(\lambda)|\Omega|_{P_{\mathcal{G}}}$ agrees with $J_G(\lambda)\prod_{i\in G}\mathrm{d}^D k_i$ on $E_G((\mathbb{R}^*)^D)\times V_G^{ext}(\mathbb{R}^D)$.

Proof. Let $\overline{z}=(\overline{s},\overline{u},\overline{K})\in\mathbb{C}^{\mathcal{G}}\backslash Z_{\mathcal{G}}$ and as above let

$$\begin{aligned}
\overline{\mathcal{I}}&=\{\overline{\gamma}_1\subsetneq\ldots\subsetneq\overline{\gamma}_r\}=\{\gamma\subseteq G\mid \overline{u}_\gamma=0\},\\
\overline{\mathcal{J}}&=\{\overline{\eta}_1\subsetneq\ldots\subsetneq\overline{\eta}_t\}=\{\eta\subseteq G^0\mid \overline{s}_\eta=0\},\\
\overline{k}_i&=\left(\prod_{i\in\gamma\notin\overline{\mathcal{I}}}u_\gamma\right)^{-1}\left(\prod_{i\in\eta\notin\overline{\mathcal{J}}}s_\eta\right)K_i.
\end{aligned}$$

On a neighbourhood $\mathcal{U}_{\overline{z}}$, where s_η and u_γ are non-vanishing for $\gamma\notin\overline{\mathcal{I}}$ and $\eta\notin\overline{\mathcal{J}}$, we can use $(s,u,\overline{k})$ as coordinates. Choosing a spanning tree T^∞ of G^∞ as in the proof of Prop. 6.3.9, we can express $\overline{J}_G$ on $\mathcal{U}_{\overline{z}}\times V_G^{ext}(\mathbb{R}^D)$ as

$$\overline{J}_G(\lambda)=\prod_{i\in T_1}\delta^D(\overline{k}_i-\overline{f}_i)\prod_{b\in V_2}\delta^D(p_b-\overline{g}_b)\prod_{j\in G}(\overline{k}_j-\overline{U}_j^2m_j^2+i0)^{\lambda_j}\prod_{\gamma\subseteq G}|u_\gamma|^{\mu_\gamma}\prod_{\eta\subseteq G^0}|s_\eta|^{\nu_\eta}$$

where

$$\begin{aligned}
\mu_\gamma&=2\lambda_\gamma-Dh^1(\gamma)-1,\quad \gamma\in\overline{\mathcal{I}},\\
\nu_\eta&=D\overline{h}^1(\eta)-2\lambda_\eta-1,\quad \eta\in\overline{\mathcal{J}},\\
\mu_\gamma&=-D|\gamma|-1,\quad \gamma\notin\overline{\mathcal{I}},\\
\nu_\epsilon&=D|\eta|-1,\quad \eta\notin\overline{\mathcal{J}}.
\end{aligned}$$

and

$$\begin{aligned}
\overline{f}_i&=\sum_{j\in\overline{\eta}_t\backslash(\overline{\eta}_t\cap T_1)}C_{ji}\overline{S}_{ji}\overline{k}_j,\quad i\in T_1\cap\overline{\eta}_t,\\
\overline{f}_i&=\sum_{j\in\overline{\eta}_t\backslash(\overline{\eta}_t\cap T_1)}C_{ji}\overline{S}_j\overline{k}_j+\sum_{j\notin\gamma_r\cup\overline{\eta}_t\cup T_1}C_{ji}\overline{k}_j,\quad i\in T_1\backslash T_1\cap(\overline{\gamma}_r\cup\overline{\eta}_t),\\
\overline{g}_b&=\sum_{j\in\overline{\eta}_t\backslash(\overline{\eta}_t\cap T_1)}C_{je_b}\overline{S}_j\overline{k}_j+\sum_{j\notin\overline{\gamma}_r\cup\overline{\eta}_t\cup T_1}C_{je_b}\overline{k}_j+\sum_{a\in V_G^{ext}\backslash V_2}C_{e_ae_b}p_a,\quad b\in V_2,\\
\overline{f}_i&=\overline{U}_i\left(\sum_{j\in\overline{\eta}_t\backslash(\overline{\eta}_t\cap T_1)}C_{ji}\overline{S}_j\overline{k}_j+\sum_{j\notin\overline{\gamma}_r\cup\overline{\eta}_t\cup T_1}C_{ji}\overline{k}_j+\sum_{a\in V_G^{ext}\backslash V_2}C_{e_ai}p_a\right)\\
&\quad+\sum_{j\in\overline{\gamma}_r\backslash(\overline{\gamma}_r\cap T_1)}C_{ji}\overline{U}_{ji}\overline{k}_j,\quad i\in T_1\cap\overline{\gamma}_r,
\end{aligned}$$

with

$$\overline{S}_j = \prod_{j\in\overline{\eta}\in\overline{\mathcal{J}}} s_{\overline{\eta}}, \quad \overline{S}_{ji} = \prod_{\substack{j\in\overline{\eta}\in\overline{\mathcal{J}}\\ i\notin\overline{\eta}}} s_{\overline{\eta}}, \quad \overline{U}_j = \prod_{j\in\overline{\gamma}\in\overline{\mathcal{I}}} u_{\overline{\gamma}}, \quad \overline{U}_{ji} = \prod_{\substack{\overline{\gamma}\in\overline{\mathcal{I}}\\ i\in\overline{\gamma}, j\notin\overline{\gamma}}} u_{\overline{\gamma}}.$$

We can choose a nested set $\mathcal{N} = (\mathcal{I}, \mathcal{J}, \beta)$, such that $\mathcal{U}_{\overline{z}} \subseteq \mathcal{U}_{\mathcal{N}}$ and $\overline{\gamma}_r$ is the maximal element of $\mathcal{I}$. It follows from the coordinate description of Lemma 6.3.8 that on $\mathcal{U}_{\overline{z}}$, either $\overline{k}_j \neq 0$ or $j \notin \overline{\gamma}_r \cup G^0$. It then follows as in Prop. 6.1.1 and 6.1.2 that the product

$$J^0(z,p) = \prod_{j\in G} (\overline{k}_j - \overline{U}_j^2 m_j^2 + i0)^{\lambda_j} \prod_{\gamma\subseteq G} |u_\gamma|^{\mu_\gamma} \prod_{\eta\subseteq G^0} |s_\eta|^{\nu_\eta}$$

is well-defined and $(\overline{z}, p, i\zeta) \in SS(J^0)$ implies

$$\zeta = \sum_{j\in G} \alpha_j \overline{k}_j \cdot \mathrm{d}\overline{k}_j + \sum_{\overline{\gamma}\in\overline{\mathcal{I}}} \rho_{\overline{\gamma}} \mathrm{d}u_{\overline{\gamma}} + \sum_{\overline{\eta}\in\overline{\mathcal{J}}} \sigma_{\overline{\eta}} \mathrm{d}s_{\overline{\eta}}$$

with $\rho_{\overline{\gamma}}, \sigma_{\overline{\eta}} \in \mathbb{R}$ and $\alpha \in E_G(\mathbb{R}_{\geq 0})$ satisfying

$$\alpha_j(\overline{k}_j^2 - \overline{U}_j^2 m_j^2) = 0, \quad j \in G.$$

Now suppose we also have

$$(\overline{z}, p, i\zeta) \in SS\left(\prod_i \delta^D(\overline{k}_i - \overline{f}_i) \prod_b \delta^D(p_b - \overline{g}_b)\right)$$

i.e. there are $v \in E_{T^\infty}(\mathbb{R}^D)$ such that

$$\zeta = \sum_{i\in T_1} v_i \cdot (\mathrm{d}\overline{k}_i - \mathrm{d}\overline{f}_i(\overline{z}, p)) + \sum_{b\in V_2} v_b \cdot (\mathrm{d}p_b - \mathrm{d}\overline{g}_b(\overline{z}, p))$$

and $\overline{k}_i - \overline{f}_i(\overline{z}, p) = p_b - g_b(\overline{z}, p) = 0$. To prove that $\overline{J}_G(\lambda)$ is well-defined on $\mathcal{U}_{\overline{z}}$, it is enough to show that this implies $v = 0$.

At $\overline{z}$, we have $\overline{U}_i(\overline{z}) = \overline{S}_j(\overline{z}) = 0$ as well as

$$\overline{U}_{ji}(\overline{z}) = \begin{cases} 1, & i, j \in \overline{\gamma}_k \backslash \overline{\gamma}_{k-1} \text{ for some } \overline{\gamma}_k \in \overline{\mathcal{I}} \\ 0, & \text{otherwise} \end{cases},$$

$$\overline{S}_{ji}(\overline{z}) = \begin{cases} 1, & i, j \in \overline{\eta}_l \backslash \overline{\eta}_{l-1} \text{ for some } \overline{\eta}_l \in \overline{\mathcal{J}} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the flag of subgraphs

$$\{\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_{r+t+1} = G\} := \{\emptyset \subseteq \overline{\gamma}_1 \subseteq \ldots \subseteq \overline{\gamma}_r \subseteq G\backslash\overline{\eta}_t \subseteq \ldots \subseteq G\backslash\overline{\eta}_1 \subseteq G\},$$

and let

$$\overline{G} = \bigcup_{k=1}^{r+t+1} \Gamma_k/\Gamma_{k-1} := \bigcup_{k=1}^{r+t+1} \overline{\Gamma}_k$$

We can consider $\overline{k}$ and $\alpha \cdot \overline{k}$ as elements of $E_{\overline{G}}(\mathbb{R}^D)$. Let us write $\overline{k}_{\overline{\Gamma}}$ for the momenta corresponding to the component $\overline{\Gamma} \subseteq \overline{G}$. Comparing the coefficients of $\mathrm{d}p_b$ and $\mathrm{d}\overline{k}_i$ yields $v_b = 0$ for $b \in V_2$ and $\alpha_i \overline{k}_i = v_i$ for $i \in T_1$. The later equations then gives for $j \in \Gamma_k/\Gamma_{k+1}$:

$$\alpha \overline{k}_j + \sum_{i \in (\Gamma_k \backslash \Gamma_{k-1}) \cap T_1} C_{ji} \alpha_i \overline{k}_i = 0.$$

By Cor. 6.2.7, these equations are equivalent to $\alpha \cdot \overline{k}|_{H_1(\overline{G}, \mathbb{R}^D)} = 0$ and thus $\alpha \cdot \overline{k} = \delta_{\overline{G}} x$ for some $x \in V_{\overline{G}}(\mathbb{R}^D)$.

For $\overline{\Gamma} = \overline{\gamma}_k / \overline{\gamma}_{k-1} \subseteq \overline{G}$, the equations $\overline{k}_i = \overline{f}_i(\overline{z}, p)$ imply that $\partial_{\overline{\Gamma}} \overline{k}_{\overline{\Gamma}} = 0$ and Lemma 6.3.2 then gives $\alpha_j \overline{k}_j = 0$ for $j \in \overline{G}_k$. The case $\overline{\Gamma} = (G \backslash \overline{\eta}_{k-1})/(G \backslash \overline{\eta}_k)$ follows similarly.

For $\overline{\Gamma} = G \backslash \overline{\eta}_t / \overline{\gamma}_r$, we get $\partial_{\overline{\Gamma}} k_{\overline{\Gamma}} = -p_{\overline{\Gamma}}$, where $p_{\overline{\Gamma}}$ is the image of p under the quotient map $V_{G \backslash \overline{\eta}_t}(\mathbb{R}^D) \to V_\Gamma(\mathbb{R}^D)$. The choice of T^∞ also implies that

$$\tilde{T}^\infty = (T_1 \backslash \eta_T)/(T_1 \cap \gamma) \cup T_2$$

is a spanning tree for the completion $\overline{\Gamma}^\infty$ of $\overline{\Gamma}$. Setting $\tilde{k}_\Gamma = (\alpha \cdot \overline{k}_\Gamma, 0) \in E_{\overline{\Gamma}^\infty}(\mathbb{R}^D)$ gives $\tilde{k}_{\overline{\Gamma}}|_{H_1(\overline{\Gamma}, \mathbb{R}^D)} = 0$. Then it follows from Prop. 6.3.4, that $\delta_G(\alpha \cdot \overline{k}_\Gamma) = x_\Gamma$ for some $x_\Gamma \in V_\Gamma(\mathbb{R}^D)$ with $x_{\Gamma,a} = 0$ for $a \in V_G^{ext} \cap V_\Gamma$. From Lemma 6.3.2 we can again conclude that $\alpha_j \cdot \overline{k}_j = 0$ for $j \in \Gamma$.

Therefore $v = 0$ and $\overline{J}_G(\lambda)$ is well-defined and meromorphic. The possible poles from the factors $|u_\gamma|^{\mu_\gamma}$ and $|s_\eta|^{\nu_\eta}$ lie in the hyperplanes

$$\begin{aligned} 2\lambda_\gamma - D h^1(\gamma) &\in -\mathbb{N}, \quad \gamma \subseteq G, \\ 2\lambda_\eta - D \overline{h}^1(\eta) &\in \mathbb{N}, \quad \eta \subseteq G^0. \end{aligned}$$

Locally, we have only poles coming from γ and η in disjoint flags $\overline{\mathcal{I}}$ and $\overline{\mathcal{J}}$, which implies that every hyperplane above only contributes at most a single pole. Hence $\mathcal{J}_G(\lambda)$ has simple poles. Scaling with an element $t \in (\mathbb{C}^*)^{E_G} \times (\mathbb{C}^*)^{\mathcal{G}^*}$ gives

$$t^* \overline{\Delta}_j(u, K, \lambda) = \prod_{j \in \gamma \subseteq G} |t_\gamma|^{-2\lambda_j} \prod_{j \in \eta \subseteq G^0} |t_\eta|^{2\lambda_j} \overline{\Delta}_j(u, K, \lambda)$$

and $t^* \delta^D(d_v) = \delta^D(d_v)$. From Lemma 6.3.7, it then follows that $\overline{J}_G(\lambda)$ scales as $t^* \overline{J}_G(\lambda) = |\chi(t)| \overline{J}_G(\lambda)$, where

$$\chi = \left[\sum_{\gamma \subseteq G} (-D|\gamma| - 1) e_{F_\gamma} + \sum_{\eta \subseteq G^0} (D|\eta| - 1) e_{F_\eta} \right] = \left[- \sum_{F \in \mathcal{G}} e_F \right] = \chi_0.$$

Hence Thm. 5.6.1 gives the distributional density $\overline{J}_G(\lambda)|\Omega|_{P_\mathcal{G}}$. Using the coordinate change $k_j = \frac{S_j}{U_j} K_j$ it is easy to see that $\overline{J}_G(\lambda)|\Omega|_{P_\mathcal{G}}$ is an extension of $J_G(\lambda) \prod_{i \in G} \mathrm{d}^D k_i$. □

Definition 6.3.13. The analytically regularized Feynman amplitude of G is defined as

$$\overline{I}_G(\lambda) := \int_{P_\mathcal{G}(\mathbb{R})} \overline{J}_G(\lambda)|\Omega|_{P_\mathcal{G}}.$$

Since the support of $\overline{J}_G(\lambda)|\Omega|_{P_\mathcal{G}}$ is now proper under the projection

$$P_\mathcal{G}(\mathbb{R}) \times V_G^{ext}(\mathbb{R}^D) \to V_G^{ext}(\mathbb{R}^D)$$

we recover the following result of [SMJO76]:

Corollary 6.3.14. *The analytically regularized Feynman amplitude $\overline{I}_G$ is a well-defined, meromorphic distribution on $V_G^{ext}(\mathbb{R}^D)$ with simple poles contained in*

$$\begin{aligned} 2\lambda_\gamma - Dh^1(\gamma) &\in -\mathbb{N}, \quad \gamma \subseteq G \\ 2\lambda_\eta - D\overline{h}^1(\eta) &\in \mathbb{N}, \quad \eta \subseteq G^0. \end{aligned}$$

6.4 Microlocal Landau varieties

In this chapter we take a closer look at the singular support of the amplitude $\overline{I}_G$.

Definition 6.4.1. Let G be a connected Feynman graph. The microlocal Landau variety $\mathcal{L}_G^{\geq 0}$ is the subset of points

$$(p, i \sum_{a \in V_G^{ext}} x_a \mathrm{d}p_a) \in iT^* V_G^{ext}(\mathbb{R}^D)$$

satisfying the following condition: There are $k \in E_G(\mathbb{R}^D)$, $\tilde{x} \in V_G(\mathbb{R}^D)$ and $\alpha \in E_G(\mathbb{R}_{\geq 0})$ such that the equations

$$\begin{aligned} \partial k + p &= 0 \\ \alpha \cdot k &= -\delta \tilde{x}, \\ \alpha_j (k_j^2 - m_j^2) &= 0, \text{ for } j \in G \\ \tilde{x}_a &= x_a, \text{ for } a \in V_G^{ext} \end{aligned}$$

are satisfied.

The $(+\alpha)$-microlocal Landau variety $\mathcal{L}_G^+$ is the subset of $\mathcal{L}_G^{\geq 0}$, where the above equations are satisfiable with $\alpha \in E_G((0,\infty))$. We denote by $L_G^{\geq} = \mathring{\pi}(\mathcal{L}_G^{\geq})$ and $L_G^+ = \mathring{\pi}(\mathcal{L}_G^+)$ the corresponding projections, where

$$\mathring{\pi} : iT^* V_G^{ext}(\mathbb{R}^D) \backslash iT_M^* M \to V_G^{ext}(\mathbb{R}^D).$$

Remark 6.4.2. Let G^{ext} be the same graph as G, except that we consider all vertices external, i.e. $V_{G^{ext}}^{ext} = V_G$. The natural inclusion $i : V_G^{ext}(\mathbb{R}^D) \hookrightarrow V_{G^{ext}}^{ext}(\mathbb{R}^D) = V_G(\mathbb{R}^D)$ is defined by setting $p_v = 0$ for $v = V_G \backslash V_G^{ext}$. Under the pullback

$$i^* : iT^*(V_{G^{ext}}^{ext}(\mathbb{R}^D)) \to T^*(V_G^{ext}(\mathbb{R}^D)),$$

we have $i^* \mathcal{L}_{G^{ext}}^{\geq 0} = \mathcal{L}_G^{\geq 0}$ and $i^* \mathcal{L}_{G^{ext}}^+ = \mathcal{L}_G^+$.

Let $\gamma \subseteq G$ be a subgraph and $Q_\gamma : V_G(\mathbb{R}^D) \to V_{G/\gamma}(\mathbb{R}^D)$ be the natural projection. This restricts to a projection $Q_\gamma : V_G^{ext}(\mathbb{R}^D) \to V_{G/\gamma}^{ext}(\mathbb{R}^D)$. Under the natural duality $(V_G^{ext}(\mathbb{R}^D))^* \cong V_{G/\gamma}^{ext}(\mathbb{R}^D)$, we also have the inclusion $Q_\gamma^* : V_{G/\gamma}^{ext}(\mathbb{R}^D) \to V_G^{ext}(\mathbb{R}^D)$.

Let us write (somewhat abusively) $\mathcal{L}_{G/\gamma}^+ \subseteq \mathcal{L}_G^{\geq 0}$ for the subset

$$\{(p, i\xi) \in iT^*V_G^{ext}(\mathbb{R}^D) \mid \xi = Q_\gamma^*(\xi_q) \text{ for some } (Q_\gamma(p), i\xi_q) \in \mathcal{L}_{G/\gamma}^+\}.$$

Lemma 6.4.3. *The microlocal Landau variety can be expressed as*

$$\mathcal{L}_G^{\geq 0} = \bigcup_{\gamma \subseteq G} \mathcal{L}_{G/\gamma}^+.$$

Proof. For $(p, i\xi) \in \mathcal{L}_G^{\geq 0}$ with $\xi = \sum_{a \in V_G^{ext}} x_a \cdot \mathrm{d}p_a$, let $k \in E_G(\mathbb{R}^D), \tilde{x} \in V_G(\mathbb{R}^D)$ and $\alpha \in E_G(\mathbb{R}_{\geq 0})$ be such that the defining equations above are satisfied. Define the edge-subgraph γ by $E_\gamma = \{j \in E_G \mid \alpha_j = 0\}$.

We have the dual commutative diagrams

$$\begin{tikzcd} E_G(\mathbb{R}^D) \arrow[r, "\partial_G"] \arrow[d] & V_G(\mathbb{R}^D) \arrow[d, "Q_\gamma"] \\ E_{G/\gamma}(\mathbb{R}^D) \arrow[r, "\partial_{G/\gamma}"] & V_{G/\gamma}(\mathbb{R}^D) \end{tikzcd} \qquad \begin{tikzcd} E_G(\mathbb{R}^D) & V_G(\mathbb{R}^D) \arrow[l, "\delta_G"'] \\ E_{G/\gamma}(\mathbb{R}^D) \arrow[u] & V_{G/\gamma}(\mathbb{R}^D) \arrow[l, "\delta_{G/\gamma}"'] \arrow[u, "Q_\gamma^*"'] \end{tikzcd}$$

Since $\delta_G(\tilde{x})_j = 0$ for any edge $j \in \gamma$, we have $\tilde{x}_v = \tilde{x}_w$ for any two vertices lying in the same connected component of γ. Hence there is $\tilde{x}_q \in V_{G/\gamma}(\mathbb{R}^D)$ with $\tilde{x} = Q_\gamma^* \tilde{x}_q$. Letting k_q be the image of k under the projection $E_G(\mathbb{R}^D) \to E_{G/\gamma}(\mathbb{R}^D)$, we have

$$\begin{aligned} \partial_{G/\gamma} k_q &= Q_\gamma(p) \\ \alpha \cdot k_q &= \delta_{G/\gamma} \tilde{x}_q \\ \alpha_j(k_j^2 - m_j^2) &= 0, \quad j \in G/\gamma. \end{aligned}$$

Defining $\xi_q = \sum_{a \in V_{G/\gamma}^{ext}} \tilde{x}_a \mathrm{d}p_a$, gives $\xi = Q_\gamma^*(\xi_q)$ with $(Q_\gamma(p), \xi_q) \in \mathcal{L}_{G/\gamma}^+$. Thus $\mathcal{L}_G^{\geq 0} \subseteq \bigcup_{\gamma \subseteq G} \mathcal{L}_{G/\gamma}^+$.

Conversely, if $p_q, \tilde{x}_q \in V_{G/\gamma}^{ext}(\mathbb{R}^D), k_q \in E_{G/\gamma}(\mathbb{R}^D)$ and $\alpha \in E_{G/\gamma}((0, \infty))$ are such that the equations for $\mathcal{L}_{G/\gamma}^+$ are satisfied, and $k \in E_G(\mathbb{R}^D)$ is any lift of k_q, then setting $p = \partial_G k, \tilde{x} = Q_\gamma^* \tilde{x}_q$ and $\alpha_j = 0$ for $j \in \gamma$ gives a solution to the equations for $\mathcal{L}_G^{\geq 0}$. □

Now let $\eta \subseteq G^0$ be a subgraph consisting of massless edges. Recall that we set $V_G = V_{G \setminus \eta}$ and $V_G^{ext} = V_{G\setminus\gamma}^{ext}$. As above,we obtain

$$\mathcal{L}_{G\setminus\eta}^{\geq 0} = \bigcup_{\gamma \subseteq G\setminus\eta} \mathcal{L}_{(G\setminus\eta)/\gamma}^+.$$

We then recover the following result of [SMJO76].

Theorem 6.4.4. *The singular support of the analytically regularized Feynman integral satisfies*

$$SS(\overline{I}_G(\lambda)) \subseteq \bigcup_{\substack{\gamma\subseteq G,\eta\subseteq G^0 \\ E_\gamma\cap E_\eta=\emptyset}} \mathcal{L}^+_{G\backslash\eta/\gamma}.$$

Proof. By Lemma 6.4.3 it suffices to show that

$$SS(\overline{I}_G(\lambda)) \subseteq \bigcup_{\eta\subseteq G^0} \mathcal{L}^{\geq 0}_{G\backslash\eta}.$$

From Prop. 5.6.5 we know that $SS(\overline{I}_G)$ is contained in the subset

$$\{(p, i\xi) \in iT^*(V_G^{ext}(\mathbb{R}^D)) \mid (\overline{z}, p, 0, i\xi) \in SS(\overline{J}_G(\lambda)) \text{ for some } \overline{z} \in \mathbb{R}^{\mathcal{G}}\backslash Z_{\mathcal{G}}(\mathbb{R})\}.$$

Let $\xi = \sum_{a\in V_G^{ext}} x_a \cdot \mathrm{d}p_a$ and fix $(\overline{z}, p, 0, i\xi) \in SS(J_G(\lambda))$ with

$$\overline{z} = (\overline{s}, \overline{u}, \overline{K}) \in \mathbb{R}^{\mathcal{G}}\backslash Z_{\mathcal{G}}(\mathbb{R}).$$

Let $T^\infty = T_1 \cup T_2, \overline{f}_i, \overline{g}_b$ and $\overline{k}$ be as in the proof of Thm. 6.3.12. It follows from the local expression of $\overline{J}_G(\lambda)$ and Cor. 5.4.6 that there are $\alpha \in E_G(\mathbb{R}_{\geq 0})$ and $v \in T^\infty(\mathbb{R}^D)$, such that

$$\xi = \sum_{j\in G} \frac{1}{2}\alpha_j \mathrm{d}(\overline{k}_j^2 - \overline{U}_j^2 m_j^2) + \sum_{i\in T_1} v_i \cdot (\mathrm{d}\overline{k}_i - \mathrm{d}f_i(\overline{z}, p)) + \sum_{b\in V_2} v_b \cdot (\mathrm{d}p_b - \mathrm{d}g_b(\overline{z}, p))$$

and the following conditions are satisfied:

$$\begin{aligned} \alpha_j(\overline{k}_j - \overline{U}_j m_j^2) = 0, &\quad j \in E_G, \\ f_i(\overline{z}, k) = \overline{k}_i, &\quad i \in T_1, \\ g_b(\overline{z}, k) = p_b, &\quad b \in V_2. \end{aligned}$$

Comparing coefficients of $\mathrm{d}\overline{k}_i$ and $\mathrm{d}p_b$ for $i \in T_1$ and $b \in V_2$ gives

$$\begin{aligned} \alpha_i \overline{k}_i = -v_i, &\quad i \in T_1, \\ x_b = v_b, &\quad b \in V_2. \end{aligned}$$

Let $\Gamma = (G\backslash\overline{\eta}_t)/\overline{\gamma}_r$ and $\tilde{T}_1 = \Gamma \cap T_1/\gamma \cap T_1$. Note that $\tilde{T}_1$ intersects every component of Γ in a spanning tree, since T_1 is adapted to γ_r and $G\backslash\eta_t$. We also have $V_2 \subseteq V_\Gamma$ by the choice of T^∞. Hence $\tilde{T}^\infty = \tilde{T} \cup T_2$ is a spanning tree for the completion Γ^∞. Comparing the coefficients of $\mathrm{d}\overline{k}_j$ and $\mathrm{d}p_a$ for the variables not contained in T^∞ then gives the equations:

$$
\begin{aligned}
\overline{k}_i &= \sum_{j\in\Gamma\backslash\tilde{T}_1} C_{ji}\overline{k}_j, \quad i\in\tilde{T}_1,\\
p_b &= \sum_{j\in\Gamma\backslash\tilde{T}_1} C_{je_b}\overline{k}_j + \sum_{a\in V_G^{ext}\backslash V_2} C_{e_ae_b}p_a, \quad b\in V_2,\\
\alpha_j\overline{k}_j &= -\sum_{i\in\Gamma\cap\tilde{T}_1} C_{ji}\alpha_i k_i + \sum_{b\in V_2} C_{je_b}x_b, \quad j\in\Gamma\backslash\tilde{T}_1,\\
x_a &= -\sum_{b\in V_2} C_{e_ae_b}x_b, \quad a\in V_\Gamma^{ext}\backslash V_2,
\end{aligned}
$$

Let $\overline{k}_\Gamma = (\overline{k}_j)_{j\in\Gamma} \in E_\Gamma(\mathbb{R}^D)$. By Cor. 6.2.7 and Prop. 6.3.4, these equations are equivalent to

$$\partial_\Gamma\overline{k}_\Gamma + Q_\gamma(p) = 0, \quad (\alpha\cdot\overline{k}_\Gamma) = \delta_\Gamma\tilde{x}_\Gamma$$

for some $\tilde{x}_\Gamma \in V_\Gamma(\mathbb{R}^D)$ with $\tilde{x}_{\Gamma,a} = x_a, a\in V_\Gamma^{ext}$. Additionally we have

$$\alpha_j(\overline{k}_j^2 - m_j^2) = 0,$$

for $j\in\Gamma$ since $\overline{U}_j = 1$ for $j\notin\gamma$. Thus we obtain a solution $(\alpha,k,p,\tilde{x})$ for the defining equations of $\mathcal{L}_\Gamma^{\geq 0} \subseteq \mathcal{L}_{G\backslash\eta}^{\geq 0}$. □

The Landau varieties are quite difficult to compute explicitly. We refer to ([SMJO76], [ELOP66], [SW71]) for a detailed study of their geometries. Let us just recall the following result from [SMJO76].

Proposition 6.4.5. *Suppose G is a massive external graph. Then there is a natural isomorphism*

$$\mathcal{L}_G^+\backslash iT_M^*M \cong \{x\in V_G(\mathbb{R}^D) \mid (\delta x_i)^2 > 0 \text{ for all } i\in G\},$$

given by $x\mapsto(x,p)$, where

$$p = \partial\sum_{i\in G}\frac{m_i}{\sqrt{(\delta x_i)^2}}\delta x_i e_i$$

Proof. Given x satisfying $(\delta x_i)^2 > 0$ for all $i\in G$, let

$$k_i = -\frac{m_i}{\sqrt{(\delta x_i)^2}}\delta x_i, \quad \alpha_i = \frac{\sqrt{(\delta x_i)^2}}{m_i}.$$

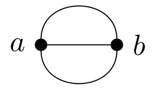
Then (x,p,k,α) satisfy the above conditions and thus $(p, i\sum_a x_a\cdot \mathrm{d}p_a)\in\mathcal{L}_G^+$.

On the other hand, if G is a massive, external graph and $(p, i\sum_a x_a\cdot\mathrm{d}p_a)\in\mathcal{L}_G^+$, then $(\delta x_i)^2 = \frac{m_i^2}{\alpha_i^2} > 0$, so that the projection

$$
\begin{aligned}
&\mathcal{L}_G^+ \to \{x\in V_G(\mathbb{R}^D) \mid (\delta x_i)^2 \neq 0 \text{ for all } i\in E_G\},\\
&(p, \sum_a x_a\cdot\mathrm{d}p_a) \mapsto (x_a)_{a\in V}
\end{aligned}
$$

is well-defined and the two maps are clearly inverse to each other. □

Example 6.4.6. Let G be a masssive graph, with two vertices a and b connected by $|G|$ edges, such as the 3-edge banana graph:



We orient all edges from a to b. Then letting $x_a = 0$, $x_b = x \in \mathbb{R}^D$ gives $k_i = \frac{m_i}{\sqrt{x^2}}$, and $\alpha_i = \frac{\sqrt{x^2}}{m_i}$. It follows that

$$p = \left(\sum_{i \in G} \frac{m_i}{\sqrt{x^2}}\right) e_b - \left(\sum_{i \in G} \frac{m_i}{\sqrt{x^2}}\right) e_a \in V_G^0(\mathbb{R}^D)$$

and therefore $p_b^2 = (\sum_{i \in G} m_i)^2$. This is defining equation for $L_G^+ \subseteq V_G^0(\mathbb{R}^D)$.

Summing the delta functions $\prod_v \delta^D(k_v - p_v)$ shows that $\overline{I}_G(\lambda)$ has support on

$$V_G^0(\mathbb{R}^D) = \{p \in V_G^{ext}(\mathbb{R}^D) \mid \sum_{a \in V_G^{ext}} p_a = 0\},$$

the space of overall momentum conservation. On the other hand, if $\eta \subseteq G^0$ is a subgraph consisting of massless edges, $p \in L_{G\backslash\eta}^{\geq 0}$ implies that $\sum_{v \in V_\gamma} p_v = 0$ for each connected component $\gamma \subseteq G\backslash\eta$. If $G\backslash\gamma$ has more than one connected component containing external momenta, then $L_{G\backslash\eta}^{\geq 0} \subseteq V_G^0$ is contained in a subspace of codimension at least D. In the sequel, we will make our life easier by working outside these special kinematic configurations.

Let $F = T\backslash j = T_1 \cup T_2$ be a spanning 2-tree, where $j \in G^0$ is a massless edge, such that both connected component of F contain external vertices. We will call such a spanning 2-tree *kinematically separating.* Let

$$L_G^F = \{p \in V_G^0(\mathbb{R}^D) \mid \sum_{a \in T_1} p_a = 0\}.$$

Definition 6.4.7. A point $p \in V_G^{ext}(\mathbb{R}^D)$ is called special, if it lies in the subset

$$L_G^s = \bigcup_F L_G^F,$$

where the the union is over all kinematically separating spanning 2-trees.

Remark 6.4.8. Let $\eta \subseteq G^0$ be a subgraph consisting of massless edges, such that $G\backslash\eta$ consists of the components $\gamma_1, \ldots, \gamma_s$. Suppose γ_1 and γ_2 contain external vertices. We can choose a spanning tree T adapted to $G\backslash\eta$ by choosing spanning trees in each component γ_i and then connecting them width edges in η. If $j \in T$ connects the components

γ_1 and γ_2, then $F = T\backslash j$ is a kinematically separating 2-tree and we have

$$\begin{aligned} V^0_{G\backslash\eta} &= \{p \in V^0_G(\mathbb{R}^D) \mid \sum_{a \in V^{ext}_G \cap \gamma_i} p_a = 0 \text{ for } i = 1, \ldots, s\} \\ &\subseteq \{p \in V^0_G(\mathbb{R}^D) \mid \sum_{a \in V^{ext}_G \cap \gamma_1} p_a = 0\} = L^F_G. \end{aligned}$$

Definition 6.4.9. A subgraph $\eta \subseteq G$ is called a *proper IR-subgraph* if $\eta \subseteq G^0$ and the complement $G\backslash\eta$ contains all external vertices in a single connected component momenta. A subgraph $\gamma \subseteq G$, such that $\eta = G\backslash\gamma$ is a proper IR-subgraph is called *mass-momentum spanning.*

Proposition 6.4.10. *On the space $V^{ext}_G(\mathbb{R}^D)\backslash L^s_G$ of nonspecial kinematics, the poles of $\overline{I}_G(\lambda)$ are contained in the set of hyperplanes*

$$\begin{aligned} &2\lambda_\gamma - Dh^1(\gamma) \in -\mathbb{N}, \quad \gamma \subseteq G \\ &2\lambda_\gamma - D\overline{h}^1(\eta) \in \mathbb{N}, \quad \eta \subseteq G^0 \textit{ proper IR-subgraph.} \end{aligned}$$

Proof. Suppose $\eta \subseteq G^0$ is not a proper IR-subgraph. Let $\overline{z} = (\overline{s}, \overline{u}, \overline{K}) \in \mathbb{R}^{\mathcal{G}}\backslash Z_{\mathcal{G}}(\mathbb{R})$ be such that $\overline{s}_\eta = 0$. We will again use the notation in the proof of Thm. 6.3.12. By possibly enlarging η, we can also assume $\eta = \overline{\eta}_t$. Let then $\Gamma = G\backslash\eta/\overline{\gamma}_r$. It follows as in loc.cit. that the momenta $\overline{k}_\Gamma$ corresponding to Γ satisfy $\partial_\Gamma \overline{k}_\Gamma = -Q_{\overline{\gamma}_r} p$.

Then we can find a kinematically separating 2-tree F, such that $p \in L^F_G$ by Remark 6.4.8. This means that, over the preimage of $V^{ext}_G(\mathbb{R}^D)\backslash L^s_G$, the support of $J_G(\lambda)|\Omega|_{P_{\mathcal{G}}}$ does not intersect the hyperplane defined by $s_\eta = 0$ and we can ignore the pole coming from the factor $|s_\eta|^{\nu_\eta}$. □

7 Parametric representation

The goal of this chapter is to express the regularized amplitude in terms of a parametric integral

$$I_G^{par}(\lambda, D) = \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(p, x) + i0} \right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}} |\Omega|_{X_{\mathcal{B}}},$$

where ψ_G and Φ_G are the first and second Symanzik polynomials associated to a Feynman graph G and $X_{\mathcal{B}}$ is a suitable blow-up of P^{E_G}. To obtain such an expression, it will be necessary to restrict to the space of non-special momenta $p \in V^0(\mathbb{R}^D) \backslash L_G^s$. As an immediate application, we can construct dimensionally regularized amplitudes in Minkowski space. In the last section, we also prove a discontinuity formula for parametric amplitudes. This chapter is based in part on the article [Sch18].

7.1 Powercounting and generalized permutahedra

We have defined the meromorphic distributions $|u_\gamma|^{\mu_\gamma}$ and $|s_\eta|^{\nu_\eta}$ appearing in $\overline{J}_G(\lambda)$ by analytic continuation. We will need to know, when they are actually given by locally integrable functions, i.e. when

$$\begin{aligned} \operatorname{Re} \lambda_\gamma &> \frac{D}{2} h^1(\gamma), \quad \gamma \subseteq G \\ \operatorname{Re} \lambda_\eta &< \frac{D}{2} \overline{h}^1(\eta), \quad \eta \subseteq G^0. \end{aligned}$$

For non-special momenta $p \in V^0(\mathbb{R}^D) \backslash L_G^s$, Prop. 6.4.10 shows that we can restrict the second condition to proper IR-subgraphs. With a view towards dimensional regularization, we will also allow the dimension to be complex. For $D \in \mathbb{C}$, let us then define the convergence region $\Lambda_G(D) \subseteq E_G(\mathbb{C})$ by $\lambda \in \Lambda_G(D)$ if and only if

$$\begin{aligned} \operatorname{Re} \lambda_\gamma &> \frac{\operatorname{Re} D}{2} h^1(\gamma) \quad \gamma \subseteq G \\ \operatorname{Re} \lambda_\eta &< \frac{\operatorname{Re} D}{2} \overline{h}^1(\eta) \quad \eta \subseteq G^0 \text{ proper IR subgraph.} \end{aligned}$$

Let us also define

$$\Lambda_G = \{(\lambda, D) \in E_G(\mathbb{C}) \times \mathbb{C} \mid \lambda \in \Lambda_G(D)\}$$

and for a subgraph $\gamma \subseteq G$:

$$\omega_\gamma := \sum_{j\in\gamma} \lambda_j - \frac{D}{2} h^1(\gamma),$$
$$\omega_{G/\gamma} := \sum_{j\in G/\gamma} \lambda_j - \frac{D}{2} h^1(G/\gamma),$$

It follows from the formula $h^1(G) = h^1(\gamma) + h^1(G/\gamma)$, that $\omega_G = \omega_\gamma + \omega_{G/\gamma}$.

The second condition above is then more naturally expressed in terms of the complement $\overline{\gamma} = G\backslash\eta$ of $\eta \subseteq G^0$. Let $\overline{\gamma}^\infty = G^\infty\backslash\eta$ be the graph obtained from $\overline{\gamma}$ by adjoining all edges $e_a \in G^\infty$ corresponding to external vertices $a \in V_G^{ext}$. Then we have

$$\begin{aligned}\lambda_\eta - \frac{D}{2}\overline{h}^1(\eta) &= \lambda_G - \lambda_{\overline{\gamma}} - \frac{D}{2}(h^1(G^\infty) - h^1(\overline{\gamma}^\infty)) \\ &= \lambda_G - \lambda_{\overline{\gamma}} - \frac{D}{2}(h^1(G) + |V_G^{ext}| - 1 - h^1(\overline{\gamma}^\infty/\overline{\gamma}) - h^1(\overline{\gamma})) \\ &= \omega_{G/\overline{\gamma}} - \frac{D}{2}(|V_G^{ext}| - 1 - h^1(\overline{\gamma}^\infty/\overline{\gamma})).\end{aligned}$$

Lemma 7.1.1. *With $\overline{\gamma}$ and $\overline{\gamma}^\infty$ as above, we have*

$$|V_G^{ext}| - 1 - h^1(\overline{\gamma}^\infty/\overline{\gamma}) \geq 0,$$

with equality if and only if $\overline{\gamma}$ is mass-momentum spanning.

Proof. Let $\overline{\gamma}_1, \ldots, \overline{\gamma}_k$ be the connected components of γ containing external vertices and let $V^c = V_G^{ext}\backslash(V_G^{ext} \cap V_{\overline{\gamma}})$ be the set of external vertices not contained in $\overline{\gamma}$. Then

$$V_{\overline{\gamma}^\infty/\overline{\gamma}} \cong \{v_\infty\} \sqcup V^c \sqcup \{v_{\overline{\gamma}_1}, \ldots, v_{\overline{\gamma}_k}\}$$

and $E_{\overline{\gamma}^\infty/\overline{\gamma}}$ consists of one edge connecting v_∞ to a for each element $a \in V^c$ and of n_i edges connecting v_∞ to $v_{\overline{\gamma}_i}$ for each subgraph $\overline{\gamma}_i$, where $n_i = |V_G^{ext} \cap V_{\overline{\gamma}_i}|$ is the number of external edges contained in $\overline{\gamma}_i$. Hence

$$h^1(\overline{\gamma}^\infty/\overline{\gamma}) = \sum_{i=1}^{k}(n_i - 1) = |V_G^{ext}| - |V^c| - k \leq |V_G^{ext}| - 1.$$

Equality is only possible if $|V^c| = 0$ and $k = 1$, since we assume that $|V_G^{ext}| \neq 1$. □

Hence if $\eta \subseteq G^0$ and $\overline{\gamma} = G\backslash\eta$ contains all external vertices of G in a single connected component, then

$$\operatorname{Re}\lambda_\eta - \frac{\operatorname{Re} D}{2}\overline{h}^1(\eta) < 0,$$

is equivalent to

$$\operatorname{Re}\omega_{G/\overline{\gamma}} = \operatorname{Re}(\omega_G) - \operatorname{Re}(\omega_\gamma) < 0.$$

This gives the following more concise description of $\Lambda_G(D)$.

Corollary 7.1.2. *The convergence region* $\Lambda_G(D)$ *is the set of points* $\lambda \in E_G(\mathbb{C}) \times \mathbb{C}$ *satisfying*

$$\begin{aligned}
&\operatorname{Re}\omega_G > 0,\\
&\operatorname{Re}\omega_\gamma > \operatorname{Re}\omega_G, \quad \gamma \subsetneq G \textit{ mass-momentum spanning},\\
&\operatorname{Re}\omega_\gamma > 0, \quad \gamma \subsetneq G \textit{ not mass-momentum spanning.}
\end{aligned}$$

Set $\delta^{mm}(\gamma)$ to be 1 if γ is mass-momentum spanning and 0 otherwise. Then let us consider $h^1(-)$ and $\delta^{mm}(-)$ as subset functions $E_G \to \mathbb{N}$ and define

$$z_G : E_G \to \mathbb{N}, \quad z_G(\gamma) = 2h^1(\gamma) + \delta^{mm}(\gamma).$$

Proposition 7.1.3. *Suppose* $\operatorname{Re} D > 0$. *Let* $Q(h^1), Q(\delta^{mm}), Q(z_G) \subseteq E_G(\mathbb{R})$ *be the polytopes associated to the above subset functions as in Section 3.9. Then* $\Lambda_G(D)$ *is nonempty if and only if* $\dim Q(z_G) = |G|-1$. *If this condition is satisfied, then* $\lambda \in \Lambda_G(D)$ *if and only if* $\operatorname{Re}\omega_G > 0$ *and* $\operatorname{Re}\lambda$ *lies in the relative interior of the scaled polytope*

$$Q_G(\omega, D) := \frac{\operatorname{Re} D}{2} Q(h^1) + (\operatorname{Re}\omega_G) Q(\delta^{mm}).$$

Proof. Let $(\lambda, D) \in E_G(\mathbb{C}) \times \mathbb{C}$ such that $\operatorname{Re}\omega_G > 0$. The polytope $Q_G(\omega, D) \subseteq E_G(\mathbb{R})$ is defined as the set of points $m \in E_G(\mathbb{R})$ satisfying:

$$\begin{aligned}
\langle e^G, m\rangle &= \frac{\operatorname{Re} D}{2} h^1(G) + \operatorname{Re}\omega_G = \langle e^G, \operatorname{Re}\lambda\rangle\\
\langle e^\gamma, m\rangle &\geq \frac{\operatorname{Re} D}{2} h^1(\gamma) + (\operatorname{Re}\omega_G)\delta^{mm}(\gamma)
\end{aligned}$$

Hence $\lambda \in \Lambda_G(D)$ if and only if $\operatorname{Re}\lambda$ lies in the interior of $Q_G(\omega, D)$, considered as a polytope in the affine space

$$H_G(\lambda) = \{m \in E_G(\mathbb{R}) \mid \langle e^G, m - \operatorname{Re}\lambda\rangle = 0\}.$$

This interior is nonempty if and only if $\dim Q_G(\omega, D) = \dim H_G(\lambda) = |G| - 1$. By Example 3.5.7, the polytope $Q_G(\omega, D)$ is normally equivalent to $Q(z_G)$ and thus their dimensions agree. □

Definition 7.1.4. The polytope $Q(z_G)$ associated to the subset function z_G is called the *Feynman polytope* of G and denoted by Q_G.

The following proposition connects the above considerations to the results in Section 3.9.

Proposition 7.1.5. *The function* z_G *is supermodular. Hence the Feynman polytope* $Q_G = Q(z_G)$ *is a generalized permutahedron.*

The proof needs a small technical lemma.

Lemma 7.1.6. *A subgraph $\gamma \subseteq G$ with $E^m_G \subseteq E_\gamma$ is mass-momentum-spanning if and only if, for every pair (T,i) of a spanning tree $T \subseteq G$ and an edge $i \in T$ disconnecting T into two components both containing external vertices, either $i \in \gamma$ or T is not adapted to γ.*

Proof. Let (T,i) be as above, such that $i \notin \gamma$. Then for every connected component $\gamma_0 \subseteq \gamma$, $T \cap \gamma_0 = (T \backslash i) \cap \gamma_0$ and either $T \cap \gamma_0 \subseteq \gamma_0$ is not a spanning tree, or γ_0 has trivial intersection with one of the components of $T \backslash i$ and thus can not contain all external vertices. Hence γ is either not mass-momentum spanning or T is not adapted to γ.

Conversely, suppose γ is not mass-momentum spanning. By possibly adjoining a component of $G \backslash \gamma$, we can assume that γ has at least two components γ_1 and γ_2 with external momenta. Let T be a spanning tree adapted to γ and suppose $i \in T$ is part of a bridge which connects the two components $T \cap \gamma_1$ and $T \cap \gamma_2$ of $\gamma \cap T$. Then (T,i) satisfies the hypothesis of the lemma and $i \notin \gamma$. □

Proof of Prop. 7.1.5. The function $2h^1$ is supermodular by Prop. 6.2.2. We therefore only need to show that

$$z_G(\gamma_1) + z_G(\gamma_2) \leq z_G(\gamma_1 \cup \gamma_2) + z_G(\gamma_1 \cap \gamma_2),$$

where γ_1 and γ_2 (and thus $\gamma_1 \cup \gamma_2$) are mass-momentum spanning. We can also assume that $h^1(\gamma_1) + h^1(\gamma_2) = h^1(\gamma_1 \cup \gamma_2) + h^1(\gamma_1 \cap \gamma_2)$, since the inequality is trivial otherwise and reduce to the case $\gamma_1 \cup \gamma_2 = G$.

We will show that $\gamma_1 \cap \gamma_2$ is also mass-momentum spanning. It is clear that $\gamma_1 \cap \gamma_2$ contains all massive edges. Let (T,i) be a pair as in the above lemma and suppose T is adapted to $\gamma_1 \cap \gamma_2$. Then $T \subseteq \gamma_1 \cup \gamma_2$ is a spanning tree such that $T \cap \gamma_1 \cap \gamma_2$ is a maximal forest, we must have $i \in \gamma_1 \cap \gamma_2$. By the above assumption about the loop numbers, we have

$$\begin{aligned} h^1(\gamma_1) + h^1(\gamma_2) &= h^1(\gamma_1 \cup \gamma_2) + h^1(\gamma_1 \cap \gamma_2) \\ &= |\gamma_1 \cup \gamma_2| + |\gamma_1 \cap \gamma_2| - |(\gamma_1 \cup \gamma_2) \cap T| - |\gamma_1 \cap \gamma_2 \cap T| \\ &= (|\gamma_1| - |\gamma_1 \cap T|) + (|\gamma_2| - |\gamma_2 \cap T|). \end{aligned}$$

Then T must be adapted to γ_1 and γ_2 by Prop. 6.2.5. Since both graphs are mass-momentum spanning, the above lemma shows $i \in \gamma_1$ and $i \in \gamma_2$. Then $i \in \gamma_1 \cap \gamma_2$ and $\gamma_1 \cap \gamma_2$ must also be mass-momentum spanning. □

For an m.m. subgraph γ, we set $V^{ext}_\gamma = V^{ext}_G$ and $E^m_\gamma = E^m_G$. Otherwise we consider γ to be scaleless, i.e. $V^{ext}_\gamma = E_\gamma = \emptyset$. The kinematics of G/γ are inherited from G in the obvious way. This implies that G/γ has nontrivial kinematics if and only if γ is not mass-momentum spanning. We extend the definition of z_G to graphs with trivial kinematics by setting $z_G(\gamma) = 2h^1(\gamma)$.

These conventions on the kinematics of sub- and quotient graphs are justified by the following.

Proposition 7.1.7. *The restrictions and contractions of z_G by a subgraph $\gamma \subsetneq G$ are given by*

$$z_G|_\gamma = z_\gamma, \quad z_G/_\gamma = z_{G/\gamma}.$$

Proof. The equality $z_G|_\gamma = z_\gamma$ for the restrictions follows immediately from the definitions. For $\eta \subseteq G/\gamma$, let $\tilde{\eta}$ be the edge subgraph corresponding to $E_\gamma \cup E_\eta$. The contraction equality then claims that

$$\begin{aligned} z_G/_\gamma(\eta) &= 2h^1(\tilde{\eta}) - 2h^1(\gamma) + \delta^{mm}(\tilde{\eta}) - \delta^{mm}(\gamma) \\ &\overset{!}{=} 2h^1(\eta) + \delta^{mm}(\eta) = z_{G/\gamma}(\eta). \end{aligned}$$

The equality $h^1(\tilde{\eta}) - h^1(\gamma) = h^1(\eta)$ follows directly from Prop. 6.2.2. We then only have to prove that $\tilde{\eta}$ is mass-momentum spanning if η is. Clearly η contains all massive edges of G/γ if and only if $\tilde{\eta}$ contains all massive edges of G. Similarly, if $\tilde{\eta}$ contains all external vertices in a single connected component, then so does its contraction η. On the other hand, if all external vertices of G/γ lie in the component $\eta_0 \subseteq \eta$ and $\gamma_1, \ldots, \gamma_s \subseteq \gamma$ are the components of γ containing external vertices, then the subgraph $\tilde{\eta}_0 \subseteq \tilde{\eta}$ defined by the edge set $E_{\eta_0} \cup E_{\gamma_1} \ldots \cup E_{\gamma_s}$ is connected and contains all external vertices. $\square$

Let us recall that a graph G is called *1-vertex reducible* if the removal of any vertex disconnects the graph and *1-vertex irreducible (1VI)* otherwise. We consider graphs with a single edge to be 1VI and disconnected graphs to be 1-vertex reducible. A graph on two vertices is 1VI if it contains no self-loops. Any graph then has a unique decomposition into 1VI-subgraphs.

Lemma 7.1.8. *Suppose $\gamma_1, \gamma_2 \subseteq G$ are nontrivial, edge-disjoint subgraphs, such that $G = \gamma_1 \cup \gamma_2$. Then the following are equivalent:*

1. *γ_1 and γ_2 are unions of 1VI-components of G.*
2. *For all subgraphs $\eta \subseteq G$, $h^1(\eta) = h^1(\eta \cap \gamma_1) + h^1(\eta \cap \gamma_2)$.*
3. *$h^1(G) = h^1(\gamma_1) + h^1(\gamma_2)$.*

Proof. ($1 \Rightarrow 2$) If γ_1 and γ_2 are unions of pairwise edge-disjoint 1VI components, then G is homotopy-equivalent to the wedge sum $\gamma_1 \vee \gamma_2$ and the above equality follows.

The implication ($2 \Rightarrow 3$) is trivial.

($3 \Rightarrow 1$) Let $v \in V_{\gamma_1} \cap V_{\gamma_2}$. We must show that $G \backslash v$ is disconnected. We can assume that there are vertices $v_1 \in V_{\gamma_1} \backslash V_{\gamma_2}$ and $v_2 \in V_{\gamma_2} \backslash V_{\gamma_1}$, since otherwise, one of the subgraphs would be a selfloop and thus clearly a $1VI$-component. We can also assume that v_i lies in the same connected component $\tilde{\gamma}_i$ of γ_i as v. If $G \backslash v$ were connected, we could find a loopless path $\tilde{T}$ between v_1 and v_2 which does not go through v. Complete $\tilde{T}$ to a spanning tree T of G. Then

$$
\begin{aligned}
|T \cap \gamma_1| + |T \cap \gamma_2| &= |T| = |G| - h^1(G) \\
&= |\gamma_1| - h^1(\gamma_1) + |\gamma_2| - h^1(\gamma_2)
\end{aligned}
$$

and T must be adapted to γ_1 and γ_2 by Prop. 6.2.5. Thus the intersections $T \cap \tilde{\gamma}_i$ are connected and contain v. But then there must be an additional path between v_1 and v_2 going through v, which contradicts $h^1(T) = 0$. Hence $G \backslash v$ can not be connected. □

We will call a Feynman graph G *kinematically irreducible*, if every 1VI-component γ has nontrivial kinematics. This means that either γ contains massive edges, or its removal would disconnect the graphs into two components, each containing external vertices. This is compatible with the notion of irreducibilty defined in section 3.9:

Proposition 7.1.9. *The Feynman polytope Q_G is an irreducible generalized permutahedron if and only if G is kinematically irreducible.*

Proof of Prop. 7.1.9. Suppose $Q(z_G)$ is reducible. This means there are edge disjoint subgraphs $\gamma_1, \gamma_2 \subseteq G$ such that $G = \gamma_1 \cup \gamma_2$ and for all $\eta \subseteq G$:

$$
z_G(\eta) = 2h^1(\eta) + \delta^{mm}(\eta) = 2h^1(\eta \cap \gamma_1) + 2h^1(\eta \cap \gamma_2) + \delta^{mm}(\eta \cap \gamma_1) + \delta^{mm}(\eta \cap \gamma_2).
$$

For $\eta = G$ this gives $h^1(G) = h^1(\gamma_1) + h^1(\gamma_2)$ and $1 = \delta^{mm}(\gamma_1) + \delta^{mm}(\gamma_2)$. Hence the subgraphs γ_i are unions of $1VI$ components and exactly one, say $\gamma_2 = G \backslash \gamma_1$ is mass-momentum spanning. But this means that G must be kinematically reducible.

Now suppose G is kinematically reducible, i.e. there is a 1VI-component $\gamma_1 \subseteq G$ such that $\gamma_2 = G \backslash \gamma_1$ is mass-momentum spanning. Then a subgraph $\eta \subseteq G$ is mass-momentum spanning if $\eta \cap \gamma_2$ is and the above lemma shows

$$
z_G(\eta) = 2h^1(\eta \cap \gamma_1) + 2h^1(\eta \cap \gamma_2) + \delta^{mm}(\eta \cap \gamma_2) = z|_{\gamma_1}(\eta) + z|_{\gamma_2}(\eta),
$$

which means z_G is reducible. □

We then recover the following well-known result, originally proven by Speer in [Spe75] for 1VI graphs, and extended to kinematically irreducible graphs in [Smi12].

Corollary 7.1.10. *The convergence domain $\Lambda_G(D)$ for $\operatorname{Re} D > 0$ is nonempty if and only if G is kinematically irreducible.*

Now suppose G is kinematically irreducible. Let $\mathcal{F}_G$ denote the set of all edge subgraphs $\gamma \subsetneq G$, such γ and G/γ are both kinematically irreducible, when given the kinematics described in the beginning of this section. $\mathcal{F}_G$ is the disjoint union of the two subsets

$$
\begin{aligned}
\mathcal{S}_G &= \{\gamma \subsetneq G \mid \gamma \text{ is kinematically irreducible and m.m, } G/\gamma \text{ is irreducible}\} \\
\mathcal{U}_G &= \{\gamma \subsetneq G \mid \gamma \text{ is irreducible and not m.m, } G/\gamma \text{ is kinemetically irreducible}\}.
\end{aligned}
$$

By Prop. 3.9.11, these are exactly the facets of P_G.

Corollary 7.1.11. *Let G be a kinematically irreducible Feynman graph. Then the polytope Q_G has the facet presentation*

$$Q_G = \{\langle m, e^{E_G}\rangle = 2h^1(G) + 1\} \bigcap_{\gamma \in \mathcal{F}_G} \{\langle m, e^\gamma\rangle \geq 2h^1(\gamma) + \delta^{mm}(\gamma)\}.$$

For the convergence domain $\Lambda_G(D)$ with $\operatorname{Re} D > 0$, we get

$$\Lambda_G(D) = \{\lambda \in E_G(\mathbb{C}) \mid \operatorname{Re}\omega_G > 0, \langle \operatorname{Re}\lambda, e^\gamma\rangle \geq \frac{\operatorname{Re} D}{2} h^1(\gamma) + \operatorname{Re}\omega_G \delta^{mm}(\gamma) \text{ for } \gamma \in \mathcal{F}_G\}.$$

Remark 7.1.12. Our conventions and terminology is based on [Bro17]. There are several closely related but often slightly different notions of IR-divergent subgraph in the literature ([Bro17], [Smi12], [Spe75], [SC85]), variously working with mass-momentum graphs $\gamma \subseteq G$, its complement $\eta \subseteq G^0$ or the quotient G/γ. In the terminology of [Spe75], kinematically irreducible, mass-momentum spanning subgraphs $\gamma \subseteq G$ are called links and a subgraph γ whose quotient G/γ is kinematically irreducible is called saturated. In [Smi12], kinematically irreducible graphs are called s-irreducible.

7.2 Symanzik polynomials

The parametric representation will express $\overline{I}_G$ in terms of an integral of two polynomial functions naturally associated to G.

Definition 7.2.1. The *first Symanzik* polynomial of a connected graph G is

$$\psi_G := \sum_T \prod_{i \notin T} \alpha_i,$$

where the sum is over all spanning trees of G. The *(massless) second Symanzik* polynomial of G with external momentum $p \in V_G^0(\mathbb{R}^D)$ is

$$\varphi_G(p, \alpha) = \sum_F p_F^2 \prod_{i \notin F} \alpha_i,$$

where the first sum is over spanning two-forests $F = T_1 \cup T_2$ and

$$p_F = \sum_{v \in V_{T_1}} p_v$$

is the total momentum flowing through T_1. The *full second Symanzik* polynomial of G is defined as

$$\Phi_G(p, \alpha) = \varphi_G(p, \alpha) - \left(\sum_{i \in G} \alpha_i m_i^2\right) \psi_G(\alpha)$$

Remark 7.2.2. By momentum conservation we have

$$p_{T_1}^2 = (-p_{T_2})^2 = p_{T_2}^2.$$

Hence the above definition of φ_G is unambiguous.

To every subgraph $\gamma \subseteq G$, we associate the "flat deformation"

$$G|\gamma = \gamma \cup G/\gamma,$$

where exactly one of γ and G/γ has nontrivial kinematics. For a possibly disconnected Feynman graph Γ as above, we generalize the definitions of the Symanzik polynomials as follows: If $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$ is the disjoint union of connected graphs Γ_i, then we set

$$\begin{aligned}
\psi_\Gamma &= \sum_{i=1}^{k} \psi_{\Gamma_i} \\
\varphi_\Gamma &= \sum_{i=1}^{k} \varphi_{\Gamma_i} \prod_{j \neq i} \psi_{\Gamma_j} \\
\Phi_\Gamma &= \sum_{i=1}^{k} \Phi_{\Gamma_i} \prod_{j \neq i} \psi_{\Gamma_j}.
\end{aligned}$$

Let us also call $\gamma \subseteq G$ *momentum spanning* if it contains all external vertices in a single connected component. As above, we let $\delta^m(\gamma)$ be 1 if γ is momentum spanning and 0 otherwise.

We can now prove the following crucial factorization formula, due to Francis Brown.

Proposition 7.2.3 ([Bro17]). *Let G be a connected Feynman graph and $\gamma \subseteq G$ a subgraph with connected components $\gamma_0, \ldots, \gamma_n$. Then there are polynomials $R^{\psi}_{G|\gamma}, R^{\varphi}_{G|\gamma}$ and $R^{\Phi}_{G|\gamma}$, such that*

$$\begin{aligned}
\psi_G &= \psi_{G|\gamma} + R^{\psi}_{G|\gamma} \\
\varphi_G &= \varphi_{G|\gamma} + R^{\varphi}_{G|\gamma} \\
\Phi_G &= \Phi_{G|\gamma} + R^{\Phi}_{G|\gamma}.
\end{aligned}$$

The degree $\deg_\gamma(R^{\cdot}_{G|\gamma})$ of the rest terms in the variables $(\alpha_j)_{j\in\gamma}$ satisfies

$$\begin{aligned}
\deg_\gamma(R^{\psi}_{G|\gamma}) &> \deg_\gamma(\psi_{G|\gamma}) = h^1(\gamma) \\
\deg_\gamma(R^{\varphi}_{G|\gamma}) &> \deg_\gamma(\varphi_{G|\gamma}) = h^1(\gamma) + \delta^m(\gamma) \\
\deg_\gamma(R^{\Phi}_{G|\gamma}) &> \deg_\gamma(\Phi_{G|\gamma}) = h^1(\gamma) + \delta^{mm}(\gamma)
\end{aligned}$$

Proof. For $k = 1, 2$, consider the subset $\mathcal{T}^k_\gamma$ of spanning k-trees of G, such that the intersections $T \cap \gamma_i$ are connected for all $T \in \mathcal{T}^k_\gamma$. By Prop. 6.2.5, its elements are exactly those spanning k-trees, such that $\sum_i |T \cap \gamma_i|$ is maximal. The corresponding monomial α^S for $S = G \backslash T$ is then minimal in the variables $(\alpha_i)_{i\in\gamma}$. Decomposing the sum over k-trees into a sum over $\mathcal{T}^k_\gamma$ and its complement gives the decompositions

$$\begin{aligned}
\psi_G &= \psi_\gamma \psi_{G/\gamma} + R^{\psi}_{G|\gamma} \\
\varphi_G &= \psi_\gamma \varphi_{G/\gamma} + R^{\varphi}_{G|\gamma}.
\end{aligned}$$

Note that $\psi_{G|\gamma} = \psi_\gamma \psi_{G/\gamma}$. The polynomial is $\psi_\gamma \varphi_{G/\gamma}$ is equal to $\varphi_{G|\gamma}$ if and only if γ is not momentum-spanning.

Suppose γ is momentum-spanning and all external vertices are contained in the connected component γ_0. Then the polynomial $\varphi_{G/\gamma}$ vanishes identically and we have to use a different decomposition. In this case, we define $\tilde{\mathcal{T}}^2_\gamma$ to be those spanning 2-trees F, such that $F \cap \gamma_i$ is connected for $i > 0$, $F/F \cap \gamma$ is a tree, and F splits γ_0 into two connected components. Such 2-trees can always be constructed by choosing a spanning tree adapted to γ and deleting a suitable edge of γ_0.

Splitting the sum over all 2-trees into a sum over $\tilde{\mathcal{T}}^2_\gamma$ and its complement as above gives the decomposition

$$\varphi_G = \varphi_{\gamma_0} \psi_{\gamma_1} \cdots \psi_{\gamma_k} \psi_{G/\psi} + R^\varphi_{G|\gamma} = \varphi_{G|\gamma} + R^\varphi_{G|\gamma}$$

This factorization of Φ_G follows immediately from the factorization of ψ_G and φ_G. $\square$

The parametric representation is essentially a Mellin transform of the two polynomials ψ_G and $\Phi_G(p)$. The divergence behaviour of such an integral is determined in large part by their Newton polytopes ([NP11],[BFP14],[Sch18]).

Definition 7.2.4. Suppose

$$g \in \mathbb{C}[\alpha_j^\pm \mid j \in G], \quad g(\alpha) = \sum_{m \in E_G(\mathbb{Z})} c_m \alpha^m$$

is Laurent polynomial in the α-variables. Its Newton polytope is defined as the convex hull

$$Q(g) = \mathrm{Conv}(m \mid c_m \neq 0).$$

Every monomial $\alpha^S = \prod_{i \in S} \alpha_i$ appearing in ψ_G corresponds to the complement $S = G \backslash T$ of a spanning tree $T \subseteq G$. Similarly, a monomial α^U appears in Φ_G if and only if $U = G \backslash T \cup i$, where $T \subseteq G$ is a spanning tree and $i \in G$ is either a massive edges, or $F = T \backslash i$ is a 2-tree with $p_F \neq 0$. We will call such a pair (T, i) *admissible.*

Proposition 7.2.5. *The Newton polytope of ψ_G equals the polytope of the subset function $h^1_G(\gamma) = h^1(\gamma)$. The Newton polytope of the product $\psi_G \cdot \Phi_G$ equals the Feynman polytope $Q_G = Q(z_G)$.*

Remark 7.2.6. For Φ_G we consider its Newton polytope for a fixed $p \in V^0_G(\mathbb{R}^D) \backslash L^s_G$. This is independent of the choice of p by the above description.

Proof. The vertices of $Q(\psi_G)$ correspond to the complements $S = G \backslash T$ of spanning trees $T \subseteq G$, which are the bases of the dual graph matroid associated to G. It then follows from Exam. 3.9.5, that $Q(\psi_G) = Q(h^1_G)$.

By induction over the number of edges, we can assume that $Q(\psi_\gamma \Phi_\gamma) = Q(z_\gamma)$ for each Feynman graph γ with nontrivial kinematics and $|\gamma| < |G|$.

The above factorization formula show that

$$\langle e^\gamma, m\rangle \geq 2h^1(\gamma) + \delta^{mm}(\gamma) = z_G(\gamma)$$

and $\langle e^G, m\rangle = 2h^1(G) + 1$ for all $m \in Q(\psi_G \Phi_G)$, which means that $Q(\psi_G\Phi_G) \subseteq Q_G$. But they also imply

$$F_{e^\gamma} Q(\psi_G \Phi_G(p)) = Q(\psi_{G|\gamma}\Phi_{G|\gamma})$$

If γ is mass-momentum spanning, then

$$\begin{aligned} Q(\psi_{G|\gamma}\Phi_{G|\gamma}) &= Q(\psi_\gamma \Phi_\gamma) \times Q(\psi_{G/\gamma} \cdot \psi_{G/\gamma}) \\ &= Q(z_\gamma) \times Q(z_{G/\gamma}) \\ &= F_{e^\gamma} Q_G, \end{aligned}$$

by the induction hypothesis and Prop. 3.9.7. Similarly, if γ is not mass-momentum spanning, then

$$\begin{aligned} Q(\psi_{G|\gamma}\Phi_{G|\gamma}) &= Q(\psi_\gamma \cdot \psi_\gamma) \times Q(\psi_{G/\gamma} \times \Phi_{G/\gamma}) \\ &= Q(z_\gamma) \times Q(z_{G/\gamma}) \\ &= F_{e^\gamma} Q_G. \end{aligned}$$

Hence all vertices of Q_G are contained in $Q(\psi_G\Phi_G)$, since they are all given by intersection of proper faces. This gives the reverse inclusion $Q_G \subseteq Q(\psi_G\Phi_G)$. □

Suppose G is kinematically irreducible. Then Q_G is full-dimensional in the affine hyperplane $\{m \in E_G(\mathbb{Z}) \mid \sum_{i\in G} m_i = 2h^1_G + 1\}$. Its normal fan Σ_{Q_G} is degenerate, since each cone $\sigma \in \Sigma_{Q_G}$ contains the line $\mathbb{R}e^G$. It is more natural to consider the quotient $N_G = E_G(\mathbb{Z})/\mathbb{Z}e^G$. The corresponding collection of cones

$$\Sigma_G := \{\sigma/\mathbb{R}e^G \subseteq N_G \otimes \mathbb{R} \mid \sigma \in \Sigma_{Q_G}\}$$

is then a proper fan.

From Prop. 7.1.5, we know that Q_G is a generalized permutahedron. Hence every cone $\sigma \in \Sigma_G$ is of the form

$$\sigma_{\mathcal{I}} = \operatorname{pos}([e^\gamma] \mid \gamma \in \mathcal{I})$$

for some subset $\mathcal{I} \subseteq \mathcal{F}_G$ of subgraphs, where $\mathcal{F}_G$ is the set of subgraphs defining facets of $\mathcal{F}_G$. Thus the poset $(\Sigma_{Q_G}, \preceq)$ is isomorphic to a subposet $(\mathcal{N}_G, \subseteq)$ of $(\mathcal{F}_G, \subseteq)$ through the identification $\sigma_{\mathcal{I}} \leftrightarrow \mathcal{I}$.

Let us close this section by constructing smooth refinements of the normal fan Σ_G. We will see that the corresponding toric varieties provide convenient compactifications for the integration domain of the parametric amplitude.

Let first Σ_{π_G} be the normal fan of the permutahedron π_G on the set of edges E_G. By Proposition 7.1.5 we have the following.

Proposition 7.2.7. *The fan Σ_{π_G} is a smooth refinement of Σ_G.*

The theory of section 3.9 allows us to construct more economical refinements. First consider the subset system

$$\mathcal{B}_s = \{\gamma \subsetneq G \mid \gamma \text{ is kinematically irreducible}\}.$$

A special case of Prop. 3.9.12 then gives

Proposition 7.2.8. *The set $\mathcal{B}_s$ is a building set. The corresponding fan $\Sigma_{\mathcal{B}_s}$ is a smooth refinement of Σ_G.*

Another possibility was recently introduced in [Bro17]. Let us call a subgraph $\gamma \subseteq G$ *motic* if

$$z_G(\gamma \backslash i) < z_G(\gamma)$$

for all edges $i \in E_\gamma$, i.e. deleting an edge either drops the loop number, or destroys the property of being mass-momentum spanning. Note that for massive Feynman graphs, the motic subgraphs are exactly the disjoint unions of one-particle irreducible (1PI) graphs. Let B_{motic} be the set of motic subgraphs and

$$\mathcal{B}_{motic} = \{\gamma \subsetneq G \mid \gamma \text{ motic}\} \cup \{\{i\} \mid i \in G\}.$$

Proposition 7.2.9. *The set $\mathcal{B}_{motic}$ is a building set and the corresponding fan Σ_{motic} is a smooth refinement of Σ_G.*

Proof. By [Bro17, Thm. 3.6], the union of two motic subgraphs is again motic. Hence $\mathcal{B}_{motic}$ is a building set. Since $z_G(\gamma \backslash i) = z_G(\gamma)$ implies that $z_\gamma = z_G|_\gamma$ is reducible, we must have $\mathcal{B}_s \subseteq \mathcal{B}_{motic}$. Example 3.8.3 then shows that Σ_{motic} refines Σ_s and hence Σ_G. □

Remark 7.2.10. The toric variety associated to Σ_{motic} is the iterated blowup constructed by Brown in [Bro17]. Its 1PI variant was earlier introduced by Bloch-Esnault-Kreimer ([BEK06]).

The smooth refinements considered here also naturally provide certain sector decomposition strategies, which are used for the numerical evaluation of Feynman integrals ([Hei08], [BH00]). The correspondence between sector decompositions and smooth refinements is discussed in detail in [Sch18] (see also [KU10]). The sectors corresponding to $\Sigma_{\mathcal{B}_s}$ are the Smirnov-Speer sectors considered in ([Smi12], [SS09]), while Σ_{π_G} corresponds to the classical Hepp sectors.

Remark 7.2.11. The original Speer sectors [Spe75] can also be adapted to give a smooth refinement of Σ_G ([Sch18]), which is minimal among all possible such refinement, although the cones are much harder to describe. It does not correspond to a building set in general and the corresponding toric variety is not necessarily a blowup of projective space.

7.3 Parametric amplitude

Now suppose G is kinematically irreducible and let $\mathcal{B}$ be any of the building sets constructed in the last section. Let

$$X_{\mathcal{B}} = \mathbb{C}^{\mathcal{B}} \backslash Z_{\mathcal{B}} // \mathbb{C}^* \times (\mathbb{C}^*)^{\mathcal{B}^*}$$

be the corresponding toric variety. The blow-up map $\pi_{\mathcal{B}} : X_{\mathcal{B}} \to P^{E_G}$ is given in terms of homogeneous coordinates by

$$\alpha_i = \prod_{i \in \gamma \in \mathcal{B}} x_\gamma.$$

Recall that the cones of $\Sigma_{\mathcal{B}}$ are in bijective correspondence with nested sets $\mathcal{N} \subseteq 2^{\mathcal{B}}$. For such a nested set $\mathcal{N}$, we let

$$D_{\mathcal{N}} = \{x \in \mathbb{C}^{\mathcal{B}} \backslash Z_{\mathcal{B}} \mid x_\gamma = 0 \text{ for } \gamma \in \mathcal{N}\}.$$

and $\tilde{D}_{\mathcal{N}} \subseteq X_{\mathcal{B}}$ the corresponding toric subvariety.

For an edge subgraph $\gamma \subseteq G$, we call an admissible pair (T, i) *adapted to* γ if

$$|\gamma \cap ((G \backslash T) \cup i)| = h^1(\gamma) + \delta^{mm}(\gamma),$$

i.e. if the corresponding lattice point $m_{(T,i)} = e_i + \sum_{j \notin T} e_j$ lies in the face $F_{e^\gamma} Q(\Phi_G)$.

Let $T_{\mathcal{N}}$ be the set of spanning trees adapted to each $\gamma \in \mathcal{N}$, $P^1_{\mathcal{N}}$ be the set of 2-trees $F = T \backslash i$ with $p_F \neq 0$ and such that (T, i) is an admissible pair adapted to each $\gamma \in \mathcal{N}$, and $P^2_{\mathcal{N}}$ the set of admissible pairs adapted to each $\gamma \in \mathcal{N}$, such that $m_i \neq 0$.

Remark 7.3.1. It follows from Prop. 3.5.4, that $T_{\mathcal{N}}$ and $P^1_{\mathcal{N}} \cup P^2_{\mathcal{N}}$ are always non-empty, since the cone $\sigma_{\mathcal{N}}$ corresponding to $\mathcal{N}$ is contained in a maximal cone of Σ_G.

We will write $\psi_G(x) = \pi^*_{\mathcal{B}} \psi_G(\alpha)$ and $\Phi_G(p, x) = \pi^*_{\mathcal{B}} \Phi_G(p, \alpha)$ for the pullback of the Symanzik polynomials, expressed in the homogeneous coordinates of $X_{\mathcal{B}}$.

Proposition 7.3.2. *Let $\mathcal{N} \subseteq 2^{\mathcal{B}}$ be a nested set.*

1. *The first Symanzik polynomial can be expressed as*

$$\psi_G(x) = \prod_{\gamma \in \mathcal{N}} x_\gamma^{h^1(\gamma)} \psi_{\mathcal{N}}(x),$$

where

$$\psi_{\mathcal{N}}(x) = \sum_{T \in \mathcal{T}_{\mathcal{N}}} \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma| - |\gamma \cap T|} + R^\psi_{\mathcal{N}}(x)$$

and $R^\psi_{\mathcal{N}}(x)$ is a polynomial vanishing on $D_{\mathcal{N}}$.

2. *The second Symanzik polynomial can be expressed as*

$$\Phi_G(p, x) = \prod_{\gamma \in \mathcal{N}} x_\gamma^{h^1(\gamma) + \delta^{mm}(\gamma)} \Phi_{\mathcal{N}}(p, x),$$

where

$$\Phi_{\mathcal{N}}(p,x) = \sum_{F\in P^1_{\mathcal{N}}} p_F^2 \prod_{\gamma\notin\mathcal{N}} x_\gamma^{|\gamma|-|\gamma\cap F|} - \sum_{(T,i)\in P^2_{\mathcal{N}}} m_i^2 \prod_{\gamma\notin\mathcal{N}} x_\gamma^{|\gamma|-|\gamma\cap(T\setminus i)|} + R^\Phi_{\mathcal{N}}(p,x)$$

and $R^\Phi_{\mathcal{N}}(p,x)$ *is a polynomial vanishing on* $D_{\mathcal{N}}$*.*

Proof. We can express $\psi_G(x)$ as

$$\psi_G(x) = \sum_{T\in T_{\mathcal{N}}} \prod_{\gamma\in\mathcal{B}} x_\gamma^{|\gamma|-|T\cap\gamma|} + \sum_{T\notin T_{\mathcal{N}}} \prod_{\gamma\in\mathcal{B}} x_\gamma^{|\gamma|-|T\cap\gamma|}$$

For each $T\in T_{\mathcal{N}}$ and $\gamma\in\mathcal{N}$, we have $|\gamma|-|T\cap\gamma| = h^1(\gamma)$ and for each $T\notin T_{\mathcal{N}}$ there must be at least one $\gamma\in\mathcal{N}$, such that $|\gamma|-|T\cap\gamma| > h^1(\gamma)$. Then we can factor out the product $\prod_{\gamma\in\mathcal{N}} x_\gamma^{h^1(\gamma)}$ and obtain the above formula. The argument for Φ_G is completely analogous. $\square$

For $(\lambda, D)\in E_G(\mathbb{C})\times\mathbb{C}$, let

$$J_G^{par}(\lambda,D) = \prod_{\gamma\in\mathcal{B}} \chi_+^{\lambda_\gamma-1}(x_\gamma)\left(\frac{\psi_G(x)}{\Phi_G(p,x)+i0}\right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}},$$

where we have set $\lambda_\gamma = \sum_{i\in\gamma}\lambda_i$ as above.

Theorem 7.3.3. $J_G^{par}(\lambda,D)$ *is a well-defined distribution on*

$$(\mathbb{R}^{\mathcal{B}}\setminus Z_{\mathcal{B}}(\mathbb{R}))\times(V_G^0(\mathbb{R}^D)\setminus L_G^s),$$

which descends to a well-defined distributional density on $X_{\mathcal{B}}(\mathbb{R})\times V_G^0(\mathbb{R}^D)\setminus L_G^s$*.*

Proof. Fix $\overline{x}\in(\mathbb{R}_{\geq 0})^{\mathcal{B}}\setminus Z_{\mathcal{B}}(\mathbb{R}_{\geq 0})$ and let $\mathcal{N}$ the maximal nested set, such that $\overline{x}\in D_{\mathcal{N}}$. On a neighbourhood of $\overline{x}$, we can express J_G^{par} as

$$J_G^{par}(\lambda,D) = \prod_{\gamma\in\mathcal{B}} \chi_+^{\mu_\gamma-1}(x_\gamma)\left(\frac{\psi_{\mathcal{N}}(x)}{\Phi_{\mathcal{N}}(p,x)+i0}\right)^{\omega_G} \psi_{\mathcal{N}}(x)^{-\frac{D}{2}},$$

where

$$\mu_\gamma = \begin{cases} \lambda_\gamma - \frac{D}{2}h^1(\gamma) - \omega_G\delta^{mm}(\gamma), & \gamma\in\mathcal{N} \\ \lambda_\gamma, & \gamma\notin\mathcal{N}. \end{cases}$$

Note that $\psi_{\mathcal{N}}$ is strictly positive on $\mathcal{U}_{\overline{x}}\cap(\mathbb{R}_{\geq 0})^{\mathcal{B}}$, where $\mathcal{U}_{\overline{x}}$ is a sufficiently small neighbourhood of $\overline{x}$. Thus the corresponding factor $\psi_{\mathcal{N}}^{\omega_G-\frac{D}{2}}$ is analytic there. To prove that $\prod_{\gamma\in\mathcal{N}}\chi_+^{\mu_\gamma-1}(x_\gamma)(\Phi_{\mathcal{N}}(p,x)+i0)^{-\omega_G}$ is well-defined, we must show that

$$\xi = \sigma\mathrm{d}\Phi_{\mathcal{N}}(p,\overline{x}) = \sum_{\gamma\in\mathcal{N}}\sigma_\gamma\mathrm{d}x_\gamma,$$

for $\sigma, \sigma_\gamma \in \mathbb{R}$ and $\Phi_{\mathcal{N}}(p,\overline{x}) = 0$ implies $\xi = 0$. Fix a vertex $a_0 \in V_G^{ext}$, so that the momenta p_a for $a \neq a_0$ form a basis of $V_G^0(\mathbb{R}^D)$. For a spacetime index $\beta \in \{0, \dots, D-1\}$, the polynomial $\Phi_{\mathcal{N}}$ is quadratic in $p_{a\beta}$. By comparing the coefficients of $\mathrm{d}p_{a\beta}$ and using Euler's identity for homogeneous functions we get

$$\begin{aligned}
0 &= \sigma \sum_{a \in V^{ext}\setminus\{a_0\}} p_{a\beta} \frac{\partial \Phi_{\mathcal{N}}}{\partial p_{a\beta}}(p,\overline{x}) \\
&= \pm\sigma \sum_{F \in P^1_{\mathcal{N}}} p^2_{F\beta} \prod_{\gamma \in \mathcal{N}} x_\gamma^{|\gamma|-|\gamma\cap F|} + \sigma \sum_{a \in V^{ext}\setminus\{a_0\}} p_{a\beta} \frac{\partial R^\Phi_{\mathcal{N}}}{\partial p_{a\beta}}(p,\overline{x}) \\
&= \pm\sigma \sum_{F \in P^1_{\mathcal{N}}} p^2_{F\beta} \prod_{\gamma \in \mathcal{N}} x_\gamma^{|\gamma|-|\gamma\cap F|}
\end{aligned}$$

Hence $\xi \neq 0$ implies

$$0 = \Phi_{\mathcal{N}}(p,\overline{x}) = -\sum_{(T,i)\in P^2_{\mathcal{N}}} m_i^2 \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma|-|\gamma\cap(T\setminus i)|},$$

which is only possible if $P^2_{\mathcal{N}} = \emptyset$ and $m_j = 0$ for each $F = T\setminus j \in P^1_{\mathcal{N}}$. Then each $F \in P^1_{\mathcal{N}}$ is kinematically separating and since we have assumed that $p \notin L^s$, we must either have $\xi = 0$ or $P^2_{\mathcal{N}} = \emptyset$. The later is excluded by Remark 7.3.1 and J_G^{par} is well-defined.

Let us show that the distributional density $J_G^{par}(\lambda,p)|\Omega|_{X_{\mathcal{G}}}$ is well-defined. The support of J_G^{par} is clearly contained in $\mathbb{R}^{\mathcal{B}}_{\geq 0}\setminus Z_\Sigma(\mathbb{R}_{\geq 0}) \times (V_G^0 \setminus L_G^s)$. By Prop. 5.6.6, it is enough to show that for $t \in G_{\mathcal{B}}(\mathbb{R}_{\geq 0}) \cong (0,\infty) \times (0,\infty)^{\mathcal{B}^*}$, we have

$$t^* J_G^{par} = \chi_0(t) J_G^{par},$$

where $\chi_0 = [-\sum_{\gamma\in\mathcal{B}} e^\gamma]$. From Prop. 3.8.10, we know that

$$t^*\psi_G(x) = \tilde{\chi}(t)^{h_1(G)}\psi_G(x), \quad t^*\Phi_G(p,x) = \tilde{\chi}(t)^{h_1(G)+1}\psi_G(x),$$

for $\tilde{\chi} = \left[\sum_{i\in\gamma\in\mathcal{B}} e_\gamma\right]$ and a fixed but arbitrary $i \in G$. Then we get

$$t^* J_G^{par} = \chi(t) J_G^{par},$$

where

$$\begin{aligned}
\chi &= \left[\sum_{\gamma\in\mathcal{B}} (\lambda_\gamma - 1)e_\gamma\right] - \left(\frac{D}{2}h^1(G) - \omega_G\right)\tilde{\chi} \\
&= \left[\sum_{\gamma\in\mathcal{B}} (\lambda_\gamma - 1)e_\gamma\right] - \left[\sum_{i\in G} \lambda_i \sum_{i\in\gamma\in\mathcal{B}} e_\gamma\right] \\
&= \left[\sum_{\gamma\in\mathcal{B}} -e_\gamma\right] = \chi_0
\end{aligned}$$

□

Note that the support of $J_G^{par}|\Omega|_{X_\mathcal{B}}$ is proper under the projection to $V_G^0(\mathbb{R}^D)\backslash L_G^s$, so the following definition makes sense.

Definition 7.3.4. Let G be a kinematically irreducible Feynman graph. The (analytically regularized) parametric amplitude is defined as

$$I_G^{par}(\lambda, D) = \int_{X_\mathcal{B}(\mathbb{R})} J_G^{par}(\lambda, D)|\Omega|_{X_\mathcal{B}}.$$

For a subgraph $\gamma \subseteq G$, let us define

$$\overline{\omega}_\gamma = \begin{cases} -\omega_{G/\gamma}, & \gamma \text{ mass-momentum spanning} \\ \omega_\gamma, & \gamma \text{ not mass-momentum spanning} \end{cases}$$

From the local description of J_G^{par} given above, we get the

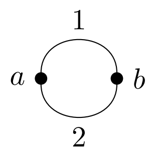
Corollary 7.3.5. *The parametric integral $I_G^{par}(\lambda, D)$ is a well-defined distribution on $V_G^0\backslash L_G^s$, meromorphic with respect to $(\lambda, D) \in E_G(\mathbb{C})\times\mathbb{C}$, and with simple poles contained in*

$$\bigcup_{\gamma\in\mathcal{B}_s} \{(\lambda, D) \in E_G(\mathbb{C}) \times \mathbb{C} \mid \overline{\omega}_\gamma \in -\mathbb{N}\}.$$

Remark 7.3.6. The above estimate on the possible poles is not quite optimal. An analogous argument, using the s-families of Speer [Spe75] (cf. Remark 7.2.11), would show that the poles are contained in the hypersurfaces

$$\bigcup_{\gamma\in\mathcal{F}_G} \{(\lambda, D) \in E_G(\mathbb{C}) \times \mathbb{C} \mid \overline{\omega}_\gamma \in -\mathbb{N}\}.$$

Example 7.3.7. Let again G be the bubble graph now without massive edges.



Let $p = p_a$ be the momentum inflowing at the left vertex. No blow-up is necessary and we can compute

$$\begin{aligned} I_G^{par}(\lambda, D) &= \int_{P^1(\mathbb{R}_{\geq 0})} \alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\left(\frac{\alpha_1+\alpha_2}{p^2\alpha_1\alpha_2+i0}\right)^{\omega_G}(\alpha_1+\alpha_2)^{-\frac{D}{2}}|\Omega|_{P^2} \\ &= (p^2+i0)^{-\omega_G}\int_{P^1(\mathbb{R}_{\geq 0})} \alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\left(\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\right)^{\omega_G}(\alpha_1+\alpha_2)^{-\frac{D}{2}}|\Omega|_{P^2} \\ &= (p^2+i0)^{-\omega_G}\int_0^1 \alpha^{\lambda_1-1-\omega_G}(1-\alpha)^{\lambda_2-1-\omega_G}\mathrm{d}\alpha \\ &= (p^2+i0)^{-\omega_G}B(\lambda_1-\omega_G, \lambda_2-\omega_G), \end{aligned}$$

where $B(x, y)$ is Euler's beta functions. The factor $(p^2 + i0)^{-\omega_G}$ is ill-defined at $p = 0$, i.e. for $p \in L^s_G$. This suggests it is possible to extend I^{par}_G to all of $V^0_G(\mathbb{R}^D)$ by judiciously blowing up subspaces in L^s_G, as we did in Section 6.1 for the massless propagator, although we will not pursue this further here.

Let us also give a more concrete representation of I^{par}_G, which will be useful to relate it to our original amplitude. For $p \in \mathbb{R}^D$ and $\theta \in [0, \frac{\pi}{2})$, we define the Wick rotation

$$p^\theta = (e^{i\theta} p_0, e^{-i\theta} p_1, \ldots, e^{-i\theta} p_{D-1})$$

Note that

$$\begin{aligned}\operatorname{Im}((p^\theta)^2) &= \sin(2\theta)\|p\|^2, \\ \operatorname{Re}((p^\theta)^2) &= \cos(2\theta) p^2.\end{aligned}$$

where $\|\cdot\|$ denotes the euclidean norm. Let

$$J^{par,\theta}_G(\lambda, D, p) = \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(p^\theta, x)}\right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}},$$

Proposition 7.3.8. *For $(\lambda, D) \in \Lambda_G$, $p \in V^0_G(\mathbb{R}^D) \backslash L^s_G$ and $\theta \in (0, \frac{\pi}{2})$ the integral*

$$I^{par,\theta}_G(\lambda, D, p) = \int_{X_{\mathcal{B}}(\mathbb{R})} J^{par,\theta}_G(\lambda, D, p) |\Omega|_{X_{\mathcal{B}}}$$

is a well-defined and absolutely convergent integral, analytic in θ, and the parametric amplitude can be expressed as the limit

$$I^{par}_G(\lambda, D) = \lim_{\theta \to 0^+} I^{par,\theta}_G(\lambda, D, p).$$

Proof. As in the proof of Thm. 7.3.3, we have the local expression in a neighbourhood of $\overline{x} \in D_{\mathcal{N}}$:

$$J^{par,\theta}_G(\lambda, D, p) = \prod_{\gamma \in \mathcal{B}} \chi_+^{\mu_\gamma - 1}(x_\gamma) \left(\frac{\psi_{\mathcal{N}}(x)}{\Phi_{\mathcal{N}}(p^\theta, x)}\right)^{\omega_G} \psi_{\mathcal{N}}(x)^{-\frac{D}{2}}$$

Note that

$$\operatorname{Im}(\Phi_{\mathcal{N}}(p^\theta, \overline{x})) = \sin(2\theta) \sum_{F \in P^1_{\mathcal{N}}} \|p_F\|^2 \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma| - |\gamma \cap F|}$$

This is strictly greater than zero for $\theta \in (0, \frac{\pi}{2})$, unless $p_F = 0$ for all $F \in P^1_{\mathcal{N}}$. But in this case

$$\Phi_{\mathcal{N}}(p^\theta, \overline{x}) = -\sum_{(T,i) \in P^2_{\mathcal{N}}} m_i^2 \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma| - |\gamma \cap (T \backslash i)|} < 0.$$

Either way, the factor $\Phi_{\mathcal{N}}(p^\theta, \overline{x})^{-\omega_G}$ is well-defined and analytic in a neighbourhood of $\overline{x}$ and depends analytically on θ. It follows from Thm. 5.3.3, that

$$\lim_{\theta \to 0^+} J_G^{par,\theta}(\lambda, D, p) = J_G^{par}(\lambda, D)$$

For $(\lambda, D) \in \Lambda_G$, the factors $\chi_+^{\mu_\gamma - 1}(x_\gamma)$ are locally integrable. Hence the function $J_G^{par}(\lambda, D, p)$ is locally integrable and of compact support, which means it is absolutely integrable. By continuity of the pushforward, we get

$$\begin{aligned} I_G^{par}(\lambda, D) &= \int_{X_{\mathcal{B}}(\mathbb{R})} J^{par}(\lambda, D)|\Omega|_{X_{\mathcal{B}}} \\ &= \lim_{\theta \to 0^+} \int_{X_{\mathcal{B}}(\mathbb{R})} J^{par,\theta}(\lambda, D, p)|\Omega|_{X_{\mathcal{B}}} \\ &= \lim_{\theta \to 0^+} I_G^{par,\theta}(\lambda, D, p). \end{aligned}$$

□

7.4 Feynman trick

Recall that our original amplitude was given by the integral of

$$J_G(\lambda, p, k) := \prod_{v \in V_G} \delta^D(p_v + k_v) \prod_{j \in G} (k_j^2 - m_j^2 + i0)^{-\lambda_j}.$$

Let

$$\tilde{J}_G(\lambda, k) = \prod_{j \in G} (k_j^2 - m_j^2 + i0)^{-\lambda_j}$$

corresponds to the propagator part of $J_G(\lambda, p, k)$.

Proposition 7.4.1. *$\tilde{J}_G(\lambda, k)$ can be expressed as the limit $\tilde{J}(\lambda, k) = \lim_{\theta \to 0^+} \tilde{J}_G^\theta(\lambda, k)$, where*

$$\tilde{J}_G^\theta(\lambda, k) = \prod_{j \in G} ((k_j^\theta)^2 - m_j^2)^{-\lambda_j}$$

Proof. For a fixed $\overline{k} \in E_G^0(\mathbb{R}^D \backslash \{0\}) \times E_G^m(\mathbb{R}^D)$, let $J = \{j \in G \mid \overline{k}_j^2 = m_j^2\}$. In a sufficiently small neighbourhood U of $\overline{k}$, we have

$$\tilde{J}_G(\lambda, k)|_U = \prod_{j \notin J} (k_j^2 - m_j^2)^{-\lambda_j} b_{\Omega_J} \left(\prod_{j \in J} (k_j^2 - m_j^2)^{-\lambda_j} \right),$$

where $\Omega_J = \bigcap_{j \in J} \{\operatorname{Im} k_j^2 > 0\}$ (see Exam. 5.4.7). Let us define the partial rotation

$$\gamma_J : [0, \frac{\pi}{2}) \times E_G(\mathbb{C}^D) \to E_G(\mathbb{C}^D)$$

by

$$\gamma_\theta(k)_j = \begin{cases} k_i^\theta, & j \in J \\ k_j, & j \notin J. \end{cases}$$

Shrinking U if necessary, we can assume that $k \neq 0$ for $j \in J$. Then

$$\mathrm{Im}(\gamma_\theta(k)_j) = \sin(2\theta)\|k_j\|^2 > 0$$

for $\theta \in (0, \frac{\pi}{2})$ and γ_J satisfies the conditions of Thm. 5.3.3. Hence we have

$$\begin{aligned} \tilde{J}_G(\lambda, k) &= \prod_{j \notin J}(k_j^2 - m_j^2)^{-\lambda_j} \lim_{\theta \to 0^+} \prod_{j \in J}((k_j^\theta)^2 - m_j^2)^{-\lambda_j} \\ &= \lim_{\theta \to 0^+} \prod_{j \in E_G}((k_j^\theta)^2 - m_j^2)^{-\lambda_j} \\ &= \lim_{\theta \to 0^+} \tilde{J}_G^\theta(\lambda, k). \end{aligned}$$

□

To resolve the product of delta functions, recall the exact sequence

$$0 \longrightarrow H_1(G, \mathbb{Z}) \xrightarrow{\ i\ } E_G(\mathbb{Z}) \xrightarrow{\ \partial\ } \tilde{V}_G^0(\mathbb{Z}) \longrightarrow 0.$$

Let $B : \tilde{V}^0(\mathbb{Z}) \to E_G(\mathbb{Z})$ be a section of ∂. This induces the isomorphism

$$E_G(\mathbb{R}^D) \cong H_1(G, \mathbb{R}^D) \oplus \tilde{V}_G^0(\mathbb{R}^D),$$

under which $\partial k + p = 0$ is equivalent to $k = l - B(p)$ for a uniquely defined loop momentum $l \in H_1(G, \mathbb{R}^D)$. For an edge $i \in G$, we will write $B(p)_i = p_i$, the choice of section being understood.

Over $p \in V_G^0(\mathbb{R}^D)$, the subvariety defined by the delta functions is the affine space

$$\partial^{-1}(-p) = B(-p) + H_1(G, \mathbb{R}^D).$$

Hence

$$\prod_{v \in V_G} \delta^D(k_v + p_v) \prod_{j \in E_G} \mathrm{d}^D k_j = \delta^D(\sum_{a \in V_G^{ext}} p_a)\chi_{H_1(G,\mathbb{R}^D)} \mathrm{d}H_G,$$

where $\mathrm{d}H_G = \prod_{j=1}^{h^1(G)} \mathrm{d}^D l_j$ is the natural Haar measure on $H_1(G, \mathbb{R}^D)$. Then our original integrand can be expressed as

$$\begin{aligned} J_G(\lambda, p, k) \prod_{j \in G} \mathrm{d}^D k_j &= \delta^D(\sum_{a \in V_G^{ext}} p_a)\chi_{H_1(G,\mathbb{R}^D)} \tilde{J}_G(\lambda, k) \mathrm{d}H_G \\ &= \delta^D(\sum_{a \in V_G^{ext}} p_a)\chi_{H_1(G,\mathbb{R}^D)} \lim_{\theta \to 0^+} \tilde{J}_G^\theta(\lambda, k) \mathrm{d}H_G \end{aligned}$$

For $\theta \in (0, \frac{\pi}{2})$ and $p \in V_G^0(\mathbb{R}^D)\backslash L_G^s$ let

$$\begin{aligned} I_G^\theta(\lambda, p) &= \int_{H_1(G,\mathbb{R}^D)} \tilde{J}_G^\theta(\lambda, l - B(p)) \mathrm{d}H_G \\ &= \int_{H_1(G,\mathbb{R}^D)} \prod_{i\in G} ((l_i^\theta - p_i^\theta)^2 - m_i^2)^{-\lambda_i} \mathrm{d}H_G \end{aligned}$$

Proposition 7.4.2. *Suppose G is kinematically irreducible and $\lambda \in \Lambda_G(D)$. Then $I_G^\theta(\lambda)$ is absolutely convergent and analytic for all $\theta \in (0, \frac{\pi}{2})$ and the Feynman amplitude $\overline{I}_G(\lambda)$ is given on $V_G^{ext}(\mathbb{R}^D)\backslash L_G^s$ as the limit*

$$\overline{I}_G(\lambda) = \delta^D\left(\sum_{a\in V_G^{ext}} p_a\right) \lim_{\theta\to 0^+} I_G^\theta(\lambda, p).$$

Proof. For $\theta \in (0, \frac{\pi}{2})$, let

$$\overline{J}_G^\theta(\lambda, p) = \prod_{i\in G}((K_i^\theta)^2 - U_i m_i^2)^{-\lambda_i} \prod_{v\in V_G} \delta^D(d_v) \prod_{\gamma\subseteq G} |u_\gamma|^{2\lambda_\gamma - D|\gamma| - 1} \prod_{\eta\subseteq G^0} |s_\eta|^{D|\eta| - 2\lambda_\eta - 1},$$

where the notation is as in Section 6.3. Then $\overline{J}_G^\theta(\lambda)$ depends analytically on θ and has the same homogeneity properties as $\overline{J}_G(\lambda)$. It follows as above that

$$\overline{J}_G(\lambda) = \lim_{\theta\to 0^+} J_G^\theta(\lambda).$$

The proof of Thm. 6.3.12 shows that we can locally express $\overline{J}_G^\theta$ as

$$\begin{aligned} \overline{J}_G^\theta(\lambda, p) &= \prod_{i\in T_1} \delta^D(\overline{k}_i - \overline{f}_i) \prod_{b\in V_2} \delta^D(p_b - \overline{g}_b) \\ &\times \prod_{j\in E_G} ((\overline{k}_j^\theta)^2 - \overline{U}_j^2 m_j^2)^{-\lambda_j} \prod_{\gamma\subseteq G} |u_\gamma|^{\mu_\gamma} \prod_{\eta\subseteq G^0} |s_\eta|^{\nu_\eta}, \end{aligned}$$

where the notations are as in loc.cit. For $p \in V_G^0(\mathbb{R}^D)\backslash L_G^s$ and $\lambda \in \Lambda_G(D)$, the factor

$$\prod_{j\in E_G} (\overline{k}_j^\theta - \overline{U}_j^2 m_j^2)^{-\lambda_j} \prod_{\gamma\subseteq G} |u_\gamma|^{\mu_\gamma} \prod_{\eta\subseteq G^0} |s_\eta|^{\nu_\eta},$$

is locally integrable and analytic outside the divisor $D_{\mathcal{G}} = \bigcup_{\gamma\subseteq G} V(u_\gamma) \cup \bigcup_{\eta\subseteq G^0} V(s_\eta)$. We know by Remark 6.3.10 that $Y_{\mathcal{G}} \cup D_{\mathcal{G}}$ is a simple normal crossing divisor. It then follows from Example 5.5.15, that $\int_{P_{\mathcal{G}}} \overline{J}_G^\theta(\lambda, p)|\Omega|_{P_{\mathcal{G}}}$ can be computed by restricting the above factor to $\overline{H}_{\mathcal{G}}\backslash\overline{H}_{\mathcal{G}} \cap \pi_{\mathcal{G}}(D_{\mathcal{G}})$ and performing the absolutely convergent integral.

Hence we get

$$\int_{P_{\mathcal{G}}} \overline{J}^{\theta}_G(\lambda, p)|\Omega|_{P_{\mathcal{G}}} = \delta^D\left(\sum_{a\in V^{ext}_G} p_a\right)\int_{H_1(G,\mathbb{R}^D)} \tilde{J}^{\theta}_G(l - B(p))\mathrm{d}H_G$$
$$= \delta^D\left(\sum_{a\in V^{ext}_G} p_a\right) I^{\theta}_G(\lambda, p).$$

Taking the limit $\theta \to 0^+$ gives

$$\overline{I}_G(\lambda) = \lim_{\theta\to 0^+}\int_{P_{\mathcal{G}}(\mathbb{R})} \overline{J}^{\theta}_G(\lambda, p)|\Omega|_{P_{\mathcal{G}}}$$
$$= \delta^D\left(\sum_{a\in V^{ext}_G} p_a\right)\lim_{\theta\to 0^+} I^{\theta}_G(\lambda, p).$$

□

The parametric representation is based on the following integral identity, due to Feynman.

Lemma 7.4.3. *Let $z = (z_1, \ldots, z_m) \in \Omega^m_+$ be a collection of elements in the upper half plane and suppose $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ satisfies $\mathrm{Re}(\lambda_i) > 0$ for $i = 1, \ldots, m$. Then we have*

$$\prod_{i=1}^m z_i^{-\lambda_i} = \Gamma(\omega)\int_{\Delta_{m-1}}\prod_{i=1}^m \frac{\alpha^{\lambda_i-1}}{\Gamma(\lambda_i)}\frac{\Omega_{\Delta_{m-1}}}{(\sum_i \alpha_i z_i)^{\omega}},$$

where $\Delta_{m-1} = \{\alpha \in \mathbb{R}^m_{\geq 0} \mid \sum_{i=1}^m \alpha_i = 1\}$ and

$$\Omega_{\Delta_{m-1}} = \sum_{i=1}^m (-1)^{i-1}\alpha_i \mathrm{d}\alpha_1 \wedge \ldots \wedge \mathrm{d}\alpha_{i-1} \wedge \mathrm{d}\alpha_{i+1} \wedge \ldots \wedge \mathrm{d}\alpha_m.$$

Proof. Both sides are well-defined and analytic for $z \in \Omega^m_+$, since the sum $\sum_i \alpha_i z_i$ has strictly positive imaginary part and thus can not vanish. Hence it is enough to prove the result for $z \in (i(0,\infty))^m \subseteq \Omega^m_+$. Factoring out the factors of i reduces to the case $z \in (0,\infty)^m$.

For $c \in (0,\infty)$ and $\mathrm{Re}(\lambda) > 0$, we have the equality

$$\int_0^\infty \alpha^{\lambda-1}e^{-t\alpha}\mathrm{d}\alpha = c^{-\lambda}\int_0^\infty u^{\lambda-1}e^{-u}\mathrm{d}u = \Gamma(\lambda)c^{-\lambda}.$$

Applying this with $c = z_i$ and $\lambda = \lambda_i$ then gives

$$\prod_{i=1}^m z_i^{-\lambda_i} = \int_{\mathbb{R}^m_{\geq 0}}\prod_{i=1}^m \frac{\alpha_i^{\lambda_i-1}}{\Gamma(\lambda_i)}e^{-\sum_i \alpha_i z_i}\mathrm{d}\alpha.$$

Let

$$\Psi : \mathbb{R}_{\geq 0} \times \Delta_{m-1} \to \mathbb{R}^m_{\geq 0}, \quad \Psi(t, \alpha) = t \cdot \alpha$$

be the natural diffeomorphism. A simple computation gives

$$\begin{aligned} \Psi^*(\mathrm{d}\alpha_1 \wedge \ldots \wedge \mathrm{d}\alpha_m) &= \mathrm{d}(t \cdot \alpha_1) \wedge \ldots \wedge \mathrm{d}(t \cdot \alpha_m) \\ &= t^{m-1} \mathrm{d}t \wedge \Omega_{\Delta_{n-1}} \end{aligned}$$

Applying the above formula now with $c = t$ and $\lambda = \omega$ gives

$$\begin{aligned} \prod_{i=1}^m z_i^{-\lambda_i} &= \int_{\Delta_{m-1}} \prod_{i=1}^m \frac{\alpha^{\lambda_i - 1}}{\Gamma(\lambda_i)} \Omega_{\Delta_{m-1}} \int_0^\infty e^{-t \sum_i \alpha_i z_i} t^{\omega - 1} \mathrm{d}t \\ &= \Gamma(\omega) \int_{\Delta_{m-1}} \prod_{i=1}^m \frac{\alpha^{\lambda_i - 1}}{\Gamma(\lambda_i)} \frac{\Omega_{\Delta_{m-1}}}{(\sum_i \alpha_i z_i)^\omega}. \end{aligned}$$

□

Let us write $\Delta_G = \Delta_{E_G - 1}$. Applying the above formula to the product $\prod_i ((k_i^\theta)^2 - m_i^2)^{-\lambda_i}$ gives the following.

Proposition 7.4.4. *For $\lambda \in \Lambda_G(D)$ and $p \in V_G^0(\mathbb{R}^D) \backslash L_G^s$, we can express $I_G^\theta(\lambda, p)$ as*

$$I_G^\theta(\lambda, p) = \Gamma(\omega) \int_{H_1(G, \mathbb{R}^D)} \int_{\Delta_G} \prod_{i \in G} \frac{\alpha_i^{\lambda_i - 1}}{\Gamma(\lambda_i)} \frac{\Omega_{\Delta_G}}{U_G(\alpha, l^\theta, p^\theta))^\omega} \mathrm{d}H_G$$

where $\omega = \sum_{i \in G} \lambda_i$ and $U_G(\alpha, l, p) = \sum_{i \in G} \alpha_i ((l_i - p_i)^2 - m_i^2)$.

For now, the above formula must be understood as an iterated integral, but we will justify exchanging the order of integration soon. To compute the integral

$$\int_{H_1(G, \mathbb{R}^D)} \frac{\mathrm{d}H_G}{U_G(\alpha, l^\theta, p^\theta))^\omega},$$

we will diagonalize the quadric

$$U_G(\alpha, l, p) = \sum_{i \in G} \alpha_i k_i^2 - \sum_{i \in G} \alpha_i m_i^2, \quad k_i = l_i - p_i$$

and relate it to to Symanzik polynomials. Our approach is based on using Plücker coordinates and is adapted from ([BEK06], [Pat10]). An alternative derivation based on the Matrix-Tree theorem can be found in ([Pan15], [BW10]).

For $\alpha \in E_G((0, \infty))$, consider the inner products

$$\begin{aligned} (\cdot, \cdot)_\alpha : E_G(\mathbb{C}) \times E_G(\mathbb{C}) \to \mathbb{C}, &\quad (k, k')_\alpha = \sum_{i \in G} \alpha_i k_i \cdot k'_i \\ \langle \cdot, \cdot \rangle_\alpha : E_G(\mathbb{C}^D) \otimes E_G(\mathbb{C}^D) \to \mathbb{C}, &\quad \langle k, k' \rangle_\alpha = \sum_{i \in G} \alpha_i k_i \cdot k'_i, \end{aligned}$$

where the product $k_i \cdot k_i'$ in the second line denotes the Minkowski inner product. The quadric can then be written as

$$U_G(\alpha, l, p) = \langle k, k \rangle_\alpha - \sum_{i \in G} \alpha_i m_i^2.$$

Suppose $\eta_1, \dots, \eta_l \in H_1(G, \mathbb{Z})$ is a $\mathbb{Z}$-basis. Let $\eta = \eta_1 \wedge \dots \wedge \eta_l \in \bigwedge^l H_1(G, \mathbb{Z})$ be the induced generator. For each subset $I = \{i_1, \dots, i_k\} \subseteq E_G$, let

$$e^I = e^{i_1} \wedge \dots \wedge e^{i_k} \in \bigwedge^k E_G(\mathbb{Z})^*,$$

where (e^i) is the natural (dual) basis of $E_G(\mathbb{Z})^*$. Note that e^I is only well-defined up to a sign, which will not matter in the sequel. The inner product $(\cdot, \cdot)_\alpha$ on $E_G(\mathbb{R}^D)$ induces an inner product on $\bigwedge^l (E_G(\mathbb{R}^D))$, which we will denote by the same symbol.

Proposition 7.4.5. *The first Symanzik polynomial of G can be expressed as*

$$\psi_G(\alpha) = \sum_{\substack{I \subseteq E_G \\ |I| = l}} (e^I(\eta))^2 \prod_{i \in I} \alpha_i = (\eta, \eta)_\alpha$$

Proof. The second equation follows immediately from the definition of $(\cdot, \cdot)_\alpha$.

The coefficient $e^I(\eta)$ vanishes or is equal to ± 1, since η is constructed in terms of a $\mathbb{Z}$-basis of the split direct summand $H_1(G, \mathbb{Z}) \subseteq E_G(\mathbb{Z})$. The term $e^I(\eta)$ is non-vanishing if and only if the composition

$$H_1(G, \mathbb{Z}) \to E_G(\mathbb{Z}) \to I(\mathbb{Z})$$

is an isomorphism. Let T be the edge subgraph defined by the complement $E_G \backslash I$. Since the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(T, \mathbb{Z}) & \longrightarrow & E_T(\mathbb{Z}) & \longrightarrow & V_T(\mathbb{Z}) & \longrightarrow & H_0(T, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & E_G(\mathbb{Z}) & \longrightarrow & V_G(\mathbb{Z}) & \longrightarrow & H_0(G, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(G/T, \mathbb{Z}) & \longrightarrow & E_{G/T}(\mathbb{Z}) & \longrightarrow & V_{G/T}(\mathbb{Z}) & \longrightarrow & H_0(G/T, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

has exact rows and columns, the above composition can only be an isomorphism if $H_1(T, \mathbb{Z}) = 0$ and $V_T(\mathbb{Z}) \to V_G(\mathbb{Z})$ is surjective, i.e. if T is a spanning tree. $\square$

Now let $p \in V_G^0(\mathbb{C}^D)$ and choose a lift $k \in E_G(\mathbb{C}^D)$, such that $\partial k = p$. Let

$$\eta(p) := \sum_{i \in G} \eta \wedge e_i k_i \in \bigwedge^{l+1} E_G(\mathbb{C}^D).$$

This does not depend on the choice of lift, since $\eta \wedge l = 0$ for each $l \in H_1(G, \mathbb{C})$. As above, there is an induced scalar product $\langle \cdot, \cdot \rangle_\alpha$. For $J \subseteq E_G$ with $|J| = l+1$, let

$$\epsilon^J = e^J \otimes id_{\mathbb{C}^D} : \bigwedge^{l+1} E_G(\mathbb{C}^D) \to \mathbb{C}^D$$

denote the natural projection.

Proposition 7.4.6. *The massless second Symanzik polynomial can be expressed as*

$$\varphi(p, \alpha) = \sum_{\substack{J \subseteq E_G \\ |J| = l+1}} (\epsilon^J(\eta(p)))^2 \prod_{j \in J} \alpha_j = \langle \eta(p), \eta(p) \rangle_\alpha$$

Proof. The second equation is again obvious from the definitions. Let $J \subseteq E_G$ with $|J| = l+1$. Standard properties of wedge products give

$$e^J(\eta(p)) = \sum_{j \in J} \pm e^{J \setminus \{j\}}(\eta) k_j.$$

We know that the RHS can only be nonvanishing if $J \setminus \{j\}$ is the complement of a spanning tree for some $j \in J$, i.e. $J = E_G \setminus F$, where $F = T_1 \cup T_2$ is a spanning 2-forest. We will show that

$$e^J(\eta(p)) = \pm p_F = \pm \sum_{v \in V_{T_1}} p_v,$$

from which the proposition follows.

Consider the contracted graph G/F. It has two vertices v_1, v_2 corresponding to the two trees T_1 and T_2. We have the commutative diagram

$$\begin{tikzcd}
H_1(G, \mathbb{C}^D) \arrow[r] \arrow[d, "\pi_J"] & E_G(\mathbb{C}^D) \arrow[r, "\partial"] \arrow[d, "\pi_J"] & V_G^0(\mathbb{C}^D) \arrow[d, "\beta"] \\
H_1(G/F, \mathbb{C}^D) \arrow[r] & J(\mathbb{C}^D) \arrow[r, "\partial"] & V_{G/F}^0(\mathbb{C}^D),
\end{tikzcd}$$

where $\pi_J : E_G(\mathbb{C}^D) \to J(\mathbb{C}^D)$ is the obvious projection and β is given by $\beta(p)_{v_i} := p_{T_i}$. The left vertical map is an isomorphism, since contracting edges of a spanning forest preserves the loop number. Hence

$$\pi_{J*}\eta = \pi_J(\eta_1) \wedge \ldots \wedge \pi_J(\eta_l)$$

is a generator of $\bigwedge^l H_1(G/F,\mathbb{Z})$ and with the above notation we can write

$$e^J(\eta(p)) = e^J(\pi_{J*}\eta(\beta(p))).$$

So we have reduced to the case of a graph G/F on two vertices, with edge set $E_{G/F} = J$ and external momenta $\beta(p)$. Let $j \in J$ correspond to an edge in G/F connecting the two vertices. Then one of $k_\pm = \pm\beta(p)_{v_1}e_j$ is a lift of of $\beta(p)$ and since the edge corresponding to j is a spanning tree, we get

$$e^J(\pi_{J*}\eta(\beta(p)))) = \pm e^{J\setminus\{j\}}\pi_{J*}\eta \cdot \beta(p)_{v_1} = \pm p_{T_1}.$$

□

Consider the $\langle\cdot,\cdot\rangle_\alpha$-orthogonal projection $\pi_\alpha : E_G(\mathbb{C}^D) \to H_1(G,\mathbb{C}^D)$. For a lift $k \in E_G(\mathbb{C}^D)$ of p, let

$$k_\alpha = k - \pi_\alpha(k).$$

This is clearly independent of the choice of k and every element $k \in \partial^{-1}(p)$ is then expressible as $k = l + k_\alpha$, where $l = \pi_\alpha(k)$ and $\langle l, k_\alpha\rangle_\alpha = 0$.

Proposition 7.4.7. *With the above notation we have*

$$\langle k_\alpha, k_\alpha\rangle_\alpha = \frac{\varphi_G(p,\alpha)}{\psi_G(\alpha)}$$

Proof. We can reduce to the case $D = 1$ by considering each component separately. In this case, we have $(\cdot,\cdot)_\alpha = \langle\cdot,\cdot\rangle_\alpha$ and it follows from Prop. 7.4.6 that

$$\begin{aligned}\varphi(\alpha,p) &= (\omega(p),\omega(p))_\alpha\\ &= (\omega\wedge k_\alpha, \omega\wedge k_\alpha)_\alpha.\end{aligned}$$

Since k_α is $(\cdot,\cdot)_\alpha$-orthogonal to $H_1(G,\mathbb{R})$ by construction, we get

$$\begin{aligned}(\omega\wedge k_\alpha, \omega\wedge k_\alpha)_\alpha &= (\omega,\omega)_\alpha(p_\alpha,p_\alpha)_\alpha\\ &= \psi(\alpha)(p_\alpha,p_\alpha)_\alpha.\end{aligned}$$

□

The main result of this section is the following.

Theorem 7.4.8. *On $V_G^0(\mathbb{R}^D)\setminus L_G^s$ we have the equality*

$$\overline{I}_G(\lambda) = \delta^D(\sum_{a\in V_G^{ext}} p_a)(-i)^{h^1(G)(D-1)}\pi^{\frac{D}{2}h^1(G)}\frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)}I_G^{par}(\lambda, D).$$

The proof needs a few preparations.

Lemma 7.4.9. *For $\lambda \in \Lambda_G(D), p \in V_G^0(\mathbb{R}^D)\backslash L_G^s$ and θ in a neighbourhood of $\theta_0 = \frac{\pi}{4}$, $I_G^\theta(\lambda, p)$ can be expressed as the absolutely convergent integral*

$$I_G^\theta(\lambda, p) = \frac{\Gamma(\omega)}{\prod_{i\in G}\Gamma(\lambda_i)} \int_{\mathring{\Delta}_G} \int_{H_1(G,\mathbb{R}^D)} \prod_{i\in G} \alpha_i^{\lambda_i - 1} \frac{\mathrm{d}H_G}{U_G(\alpha, l^\theta, p^\theta)^\omega} \Omega_{\Delta_G},$$

where $\mathring{\Delta}_G = \{\alpha \in \Delta_G \mid \alpha_i > 0, \text{ for } i \in G\}$.

Proof. Let $\lambda_i^r = \operatorname{Re}\lambda_i$ and $\omega^r = \operatorname{Re}\omega$. It is enough to show that the integral

$$F(\theta, \lambda^r, p) = \frac{\Gamma(\omega^r)}{\prod_{i\in G}\Gamma(\lambda_i^r)} \int_{\mathring{\Delta}_G} \int_{H_1(G,\mathbb{R}^D)} \prod_{i\in G} \alpha_i^{\lambda_i^r - 1} \frac{\mathrm{d}H_G}{|U_G(\alpha, l^\theta, p^\theta)|^{\omega^r}} \Omega_{\Delta_G},$$

is absolutely convergent. For θ in a neighbourhood of $\frac{\pi}{4}$, we have the estimate

$$\begin{aligned}
\sqrt{2}|U(\alpha, l^\theta, p^\theta)| &\geq |\operatorname{Im} U(\alpha, l^\theta, p^\theta)| + |\operatorname{Re} U(\alpha, l^\theta, p^\theta)| \\
&\geq |\sum_{i\in G} \alpha_i(m_i^2 - \cos(2\theta)k_i^2)| + \sin(2\theta)\sum_{i\in G}\alpha_i\|k_i\|^2 \\
&\geq \sum_{i\in G}\alpha_i(m_i^2 + (\sin(2\theta) - \cos(2\theta))\|k_i\|)^2 \\
&\geq (\sin(2\theta) - \cos(2\theta))\sum_{i\in G}\alpha_i(\|k_i\|^2 + (m_i^\theta)^2),
\end{aligned}$$

where $k_i = l_i - p_i$ and we have set $m_i^\theta = \frac{m_i}{(\sin(2\theta)-\cos(2\theta))^{\frac{1}{2}}}$. Then we get

$$\begin{aligned}
F(\theta, \lambda^r, p) &\leq 2^{\frac{\omega^r}{2}}(\sin(2\theta) - \cos(2\theta))^{-\omega^r} \frac{\Gamma(\omega^r)}{\prod_{i\in G}\Gamma(\lambda_i^r)} \\
&\quad\times \int_{H_1(G,\mathbb{R}^D)} \int_{\mathring{\Delta}_G} \prod_{i\in G} \alpha_i^{\lambda_i^r - 1} \frac{\Omega_{\Delta_G}}{(\sum_{i\in G}\alpha_i(\|l_i - p_i\|^2 + (m_i^\theta)^2))^{\omega^r}} \mathrm{d}H_G \\
&= 2^{\frac{\omega^r}{2}}(\sin(2\theta) - \cos(2\theta))^{-\omega^r} \int_{H_1(G,\mathbb{R}^D)} \prod_{i\in G}(\|l_i - p_i\|^2 + (m_i^\theta)^2)^{-\lambda_i^r}\mathrm{d}H_G \\
&= 2^{\frac{\omega^r}{2}}(\sin(2\theta) - \cos(2\theta))^{-\omega^r} \int_{H_1(G,\mathbb{R}^D)} \left|\tilde{J}_G^{\frac{\pi}{4}}(\lambda_r, l - B(p))\right| \mathrm{d}H_G,
\end{aligned}$$

and we know from Prop. 7.4.2 that the last integral is convergent. □

Lemma 7.4.10. *Let $A \in M_{N\times N}(\mathbb{C})$ be a symmetric matrix, such that $\operatorname{Im} A$ is positive definite, and $C \in \mathbb{C}$ with $\operatorname{Im}(C) > 0$. Then for $\omega \in \mathbb{C}$ with $\operatorname{Re}(\omega) > \frac{N}{2}$:*

$$\int_{\mathbb{R}^N} (x^t A x + C)^{-\omega}\mathrm{d}^N x = C^{\frac{N}{2}-\omega} \det A^{-\frac{1}{2}} \pi^{\frac{N}{2}} \frac{\Gamma(\omega - \frac{N}{2})}{\Gamma(\omega)}$$

Proof. Note that the integrand is locally integrable at $x=0$ and for $x\neq 0$ we have the bound

$$|(x^tAx+C)^{-\omega}|\leq (x^t\operatorname{Im}Ax+\operatorname{Im}C)^{-\operatorname{Re}\omega}\leq \mu_*(\operatorname{Im}A)\|x\|^{-2\operatorname{Re}(\omega)},$$

where $\mu_*(\operatorname{Im}(A))$ is the smallest eigenvalue of $\operatorname{Im}(A)$. Hence the integral is absolutely convergent and analytic in (A,C). By analytic continuation, we can reduce to the case $A=i\operatorname{Im}(A)$ and $C=i\operatorname{Im}(C)$. By an orthogonal change of basis, we can also assume that $A=i\operatorname{diag}(\mu_1,\dots,\mu_n)$ with $\mu_i\in(0,\infty)$. Then we can compute

$$\begin{aligned}\int_{\mathbb{R}^N}(x^tAx+C)^{-\omega}\mathrm{d}^Nx &= C^{-\omega}\int_{\mathbb{R}^N}\left(\sum_{j=1}^N\frac{i\mu_j}{C}x_j^2+1\right)^{-\omega}\mathrm{d}^Nx\\ &= C^{\frac{N}{2}-\omega}\prod_{j=1}^N(i\mu_j)^{-\frac{1}{2}}\int_{\mathbb{R}^N}(\|u\|^2+1)^{-\omega}\mathrm{d}^Nu\\ &= C^{\frac{N}{2}-\omega}\det(A)^{-\frac{1}{2}}\operatorname{Vol}(S^{N-1})\int_0^\infty\frac{r^{N-1}}{(r^2+1)^\omega}\mathrm{d}r\\ &= C^{\frac{N}{2}-\omega}\det(A)^{-\frac{1}{2}}\operatorname{Vol}(S^{N-1})\int_0^\infty\frac{t^{\frac{N}{2}-1}}{2(t+1)^\omega}\mathrm{d}t\\ &= C^{\frac{N}{2}-\omega}\det(A)^{-\frac{1}{2}}\operatorname{Vol}(S^{N-1})\frac{1}{2}B\left(\frac{N}{2},\omega-\frac{N}{2}\right)\end{aligned}$$

where $B(x,y)$ is Euler's Beta function. Using $\operatorname{Vol}(S^{N-1})=\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ and the well-known identity $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ gives the result. □

Lemma 7.4.11. *For $(\lambda,D)\in\Lambda_G$ and $\theta\in(0,\frac{\pi}{2})$, we have*

$$I_G^{par,\theta}(\lambda,D,p)=\int_{\mathring{\Delta}_G}\prod_{i\in G}\alpha_i^{\lambda_\gamma-1}\left(\frac{\psi_G(\alpha)}{\Phi_G(p^\theta,\alpha)}\right)^{\omega_G}\psi_G(\alpha)^{-\frac{D}{2}}\Omega_{\Delta_G}$$

Proof. Since the integral $I_G^{par,\theta}$ is absolutely convergent, we can restrict the integration domain to the dense open subset $X_{\mathcal{B}}((0,\infty))\cong P^{E_G}((0,\infty))$. The identification $P^{E_G}((0,\infty))\cong\mathring{\Delta}_G$ follows by applying the moment map of Exam. 3.6.1. □

Proof of Thm. 7.4.8. Let us first compute the integral $\int_{H_1(G,\mathbb{R}^D)}\frac{\mathrm{d}H_G}{U_G(\alpha,l^\theta,p^\theta))^\omega}$ for $\alpha\in\mathring{\Delta}_G$ and $\lambda\in\Lambda_G(D)$. The above results show that we can express the denominator in the integrand as

$$\begin{aligned}U_G(\alpha,l^\theta,p^\theta)&=\langle l^\theta-B(p^\theta),l^\theta-B(p^\theta)\rangle_\alpha-\sum_{i\in G}\alpha_im_i^2\\ &=\langle l^\theta+(k_\alpha^\theta-B(p^\theta)),l^\theta+(k_\alpha^\theta-B(p^\theta))\rangle_\alpha+\frac{\Phi_G(p^\theta,\alpha)}{\psi_G(\alpha)}.\end{aligned}$$

Shifting the integration variable $l \mapsto l + (k_\alpha - B(p))$ gives

$$\int_{H_1(G,\mathbb{R}^D)} \frac{\mathrm{d}H_G}{U_G(\alpha, l^\theta, p^\theta))^\omega} = \int_{H_1(G,\mathbb{R}^D)} \frac{\mathrm{d}H_G}{(\langle l^\theta, l^\theta\rangle_\alpha + \frac{\Phi_G(\alpha,p^\theta)}{\psi_G(\alpha)})^\omega}$$

The inner product $\langle l^\theta, l^\theta\rangle_\alpha$ can be expressed as $l^t A(\alpha,\theta) l$, where $A(\alpha,\theta) = D_1(\alpha) \otimes D_2(\theta)$ with

$$\begin{aligned} D_1(\alpha) &= \mathrm{diag}(\alpha_1, \dots, \alpha_{|G|})\big|_{H_1(G,\mathbb{R})} \\ D_2(\theta) &= \mathrm{diag}(\cos(2\theta) + i\sin(2\theta), -\cos(2\theta) + i\sin(2\theta), \dots, -\cos(2\theta) + i\sin(2\theta)) \end{aligned}$$

From Prop. 7.3.2, we have $\det(D_1(\alpha)) = \psi_G(\alpha)$. Thus the above lemma gives

$$\int_{H_1(G,\mathbb{R}^D)} \frac{\mathrm{d}H_G}{U_G(\alpha, l^\theta, p^\theta))^\omega} = \pi^{\frac{D}{2}h^1(G)} \frac{\Gamma(\omega_G)}{\Gamma(\omega)} \det(D_2(\theta))^{-\frac{h^1(G)}{2}} \psi_G^{-\frac{D}{2}} \left(\frac{\Phi_G(p^\theta,\alpha)}{\psi_G}\right)^{-\omega_G}$$

for $\theta \in (0, \frac{\pi}{2})$. Near $\theta_0 = \frac{\pi}{4}$ we obtain from the preceding lemmas, that

$$I_G^\theta(\lambda, p) = \pi^{\frac{D}{2}h^1(G)} \frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)} \det(D_2(\theta))^{-\frac{h^1(G)}{2}} I_G^{par,\theta}(\lambda, D, p).$$

Both sides are analytic in θ, so the above equality holds for all $\theta \in (0, \frac{\pi}{2})$. Combining Prop. 7.3.8 with Prop. 7.4.2 gives

$$\begin{aligned} \overline{I}_G(\lambda) &= \delta^D(\sum_{a\in V_G^{ext}} p_a) \lim_{\theta\to 0^+} I_G^\theta(\lambda, p) \\ &= \delta^D(\sum_{a\in V_G^{ext}} p_a) \pi^{\frac{D}{2}h^1(G)} \frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)} \lim_{\theta\to 0^+} \det(D_2(\theta))^{-\frac{h^1(G)}{2}} I_G^{par,\theta}(\lambda, D, p) \\ &= \delta^D(\sum_{a\in V_G^{ext}} p_a)(-i)^{h^1(G)(D-1)} \pi^{\frac{D}{2}h^1(G)} \frac{\Gamma(\omega_G)}{\prod_{i\in G}\Gamma(\lambda_i)} I_G^{par}(\lambda, D). \end{aligned}$$

□

7.5 Dimensional regularization

We can now use the preceding results to define the dimensional regularization of a Feynman integral. In dimensional regularization, one keeps the analytic parameters $\lambda \in \mathbb{C}^{E_G}$ fixed (usually at integer values) and tries to expand the above integral in a Laurent series around a point $D_0 \in \mathbb{N}$ of the spacetime dimension. There are essentially three different procedures to achieve this in the literature:

1. In the classical approach to dimensional regularization ([Col84], ['HV72]), the D-dimensional space is embedded into an infinite-dimensional space and the Feynman integral is split into a finite-dimensional subspace containing all external momenta and its orthogonal complement. Formally integrating over this infinite-dimensional complement gives an expression which is naturally analytic in the dimension D.

2. In the sector decomposition approach ([SC83], [Hei08], [BW08]), one decomposes the integration domain into cubical sectors. The ϵ-expansion is then explicitly computed in each sector by a Taylor subtraction.

3. In the analytic continuation approach ([Pan15], [vMPS15])), one extends the analytically regularized amplitude to a meromorphic function of (λ, D) and restricts to a suitable subspace (λ^0, D).

To our knowledge, there is no mathematical rigorous construction of the first approach. We refer to [Sch18] for a toric version of the sector decomposition approach, which is close to the view point adopted in this thesis. For the third approach, we actually have done most of the work already and can turn it into a definition.

Definition 7.5.1. Let G be a kinematically irreducible graph For $\lambda_0 \in E_G(\mathbb{N})$, the dimensionally regularized Feynman amplitude is the meromorphic function

$$D \mapsto I_G^{par}(\lambda^0, D),$$

with values in $Db_{V_G^0(\mathbb{R}^D)}(V_G^0(\mathbb{R}^D) \backslash L_G^s)$.

Remark 7.5.2. For Feynman graphs which are not kinematically irreducible, it is conventional to set $\overline{I}_G = 0$ in dimensional regularization. In the sector decomposition approach, these integrals are given a finite value, which luckily turns out to be zero (See [SC83] and [Sch18]).

The above definition is not very constructive, since it is not clear, how to compute the analytic continuation, when $(\lambda^0, D) \notin \Lambda_G$. Using well-known properties of Mellin transforms ([BFP14], [NP11]), one can judiciously integrate by parts and enlarge the convergence domain, until the analytic parameters inside it. The result can be summarized as follows:

Theorem 7.5.3. *The dimensionally regularized amplitude can be expanded around an integer value $D^0 \in \mathbb{N}$ of the spacetime dimension as*

$$I_G^{par}(\lambda^0, D^0 + 2\epsilon) = \sum_\beta L_\beta(\epsilon) I_G^{par}(\lambda^\beta, D^\beta + 2\epsilon) =: \sum_\beta L_\beta I_\beta$$

where $(\lambda^\beta, D^\beta) \in \mathbb{Z}^{E_G} \times \mathbb{Z}$ are shifted values of the analytic parameters and dimension and $L_\beta(\epsilon)$ are rational functions in ϵ depending polynomially on the external kinematics. Each I_β is analytic in a neighbourhood of ϵ.

We refer to [Sch18] for details and proofs (see also [vMPS15], [Pan15]).

7.6 Parametric Discontinuity formula

As an application of the preceding results, let us derive some discontinuity formula in the parametric representation. For simplicity, we restrict to massive 2-point graphs, but similar arguments work for graphs with more external vertices, when restricting to appropriate channels.

Suppose G is a massive Feynman graph, with $V_G^{ext} = \{v_a, v_b\}$. Then the parametric amplitude $I_G^{par}(\lambda, D, p) \in Db_{\mathbb{R}^D}(\mathbb{R}^D)$ is well-defined distribution in $p = p_a$. By Lorentz invariance, it only depends on the Minkowski square $s = p_a^2$, so we can write $I_G^{par}(\lambda, D, s)$. In fact, we can write the second Symanzik polynomial as $\Phi_G(p, \alpha) = \Phi_G(s, \alpha)|_{s=p_a^2}$, where

$$\Phi_G(s, \alpha) = s^2 \sum_F \prod_{i \notin F} \alpha_i - \left(\sum_{i \in G} \alpha_i m_i^2 \right) \psi_G(\alpha),$$

and the first sum is over all spanning 2-trees $F = T_1 \cup T_2$, such that both components contain one of the external vertices.

For $s \in \Omega_+$ in the upper half-plane, let

$$\begin{aligned} \tilde{I}(\lambda, D, s) &= \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(s, x)} \right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}} |\Omega|_{X_{\mathcal{B}}} \\ &=: \int_{X_{\mathcal{B}}(\mathbb{R})} \tilde{J}(\lambda, D, s) |\Omega|_{X_{\mathcal{B}}} \end{aligned}$$

It follows from Prop. 7.3.8 and Thm. 5.3.3 that $\tilde{I}(\lambda, D, s)$ is holomorphic in s and

$$I_G^{par}(\lambda, D, s) = \tilde{I}(\lambda, D, s + i0) = b_{(\operatorname{Im} s > 0)}(\tilde{I}(\lambda, D, s)),$$

Proposition 7.6.1. *There is an analytic continuation of $\tilde{I}(\lambda, D, s)$ to $\mathbb{C} \backslash \mathbb{R}_+$, such that*

$$\tilde{I}(\lambda, D, s - i0) = e^{-2\pi i \omega_G} \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(s, x) - i0} \right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}.$$

Proof. Let us define $I^-(\lambda, D, s)$ by the same equation as for $\tilde{I}(s)$ but for $\operatorname{Im}(s) < 0$. It follows analogously, that

$$I^-(\lambda, D, s - i0) = \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \left(\frac{\psi_G(x)}{\Phi_G(s, x) - i0} \right)^{\omega_G} \psi_G(x)^{-\frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}.$$

Now let $\operatorname{Re} s < 0$. Locally on $X_{\mathcal{B}}(\mathbb{R}) \times \mathbb{C}$, we can express $\tilde{J}(\lambda, D, s)$ as

$$\tilde{J}(\lambda, D, s) = \prod_{\gamma \in \mathcal{B}} \chi_+^{\mu_\gamma - 1}(x_\gamma) \left(\frac{\psi_{\mathcal{N}}(x)}{\Phi_{\mathcal{N}}(s, x)} \right)^{\omega_G} \psi_{\mathcal{N}}(x)^{-\frac{D}{2}},$$

where for $\overline{x} \in \tilde{D}_{\mathcal{N}}(\mathbb{R}_{\geq 0}) \subseteq X_{\mathcal{B}}(\mathbb{R}_{\geq 0})$ we have

$$\operatorname{Re} \Phi_{\mathcal{N}}(s, \overline{x}) = \operatorname{Re} \left(s \sum_{F \in P_{\mathcal{N}}^1} \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma| - |\gamma \cap F|} - \sum_{(T,i) \in P_{\mathcal{N}}^2} m_i^2 \prod_{\gamma \notin \mathcal{N}} x_\gamma^{|\gamma| - |\gamma \cap (T \backslash i)|} \right) < 0$$

It follows that $\tilde{J}(\lambda, D, s)$ and thus $\tilde{I}(\lambda, D, s)$ has an analytic extension to a neighbourhood of $s \in (-\infty, 0)$. The same argument shows that $I^-(\lambda, D, s)$ extends to $(-\infty, 0)$ but the extensions differ by a phase factor. Recall that we defined the branch of $z \mapsto z^{\omega_G}$ to be real-valued on $(0, \infty)$. With this convention, we have for $s \in (\infty, 0)$

$$\lim_{\epsilon \to 0^+} \tilde{I}(\lambda, D, s + i\epsilon) = e^{-2\pi i \omega_G} \lim_{\epsilon \to 0^+} I^-(\lambda, D, s - i\epsilon).$$

as in Exam. 5.5.4. □

We are interested in the jump of $\tilde{I}(\lambda, D, s)$ as s crosses the ray $\mathbb{R}_{\geq 0} \subseteq \mathbb{C}$.

Definition 7.6.2. The discontinuity $\operatorname{Disc} I^{par}(\lambda, D, s)$ is defined as the difference

$$\operatorname{Disc} I_G^{par}(\lambda, D, s) = \tilde{I}(\lambda, D, s + i0) - \tilde{I}(\lambda, D, s - i0).$$

Theorem 7.6.3. *The discontinuity* $\operatorname{Disc} I_G^{par}(\lambda, D, s)$ *can be expressed as the integral*

$$\begin{aligned}\operatorname{Disc} I_G^{par}(\lambda, D, s) = (1 - e^{-2\pi i \omega_G}) &\int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \\ &\times \left(\chi_+^{-\omega_G}(\Phi_G(s, x))\right) \psi_G(x)^{\omega_G - \frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}\end{aligned}$$

Proof. The above formulas for $\tilde{I}(s + i0)$ and $I^-(s - i0)$ give

$$\begin{aligned}\operatorname{Disc} I_G^{par}(\lambda, D, s) &= \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \psi_G(x)^{\omega_G - \frac{D}{2}} \\ &\quad \times \left(\Phi_G(s, x) + i0\right)^{-\omega_G} - e^{-2\pi i \omega_G} (\Phi_G(s, x) + i0)^{-\omega_G}\big) |\Omega|_{X_{\mathcal{B}}} \\ &= \int_{X_{\mathcal{B}}(\mathbb{R})} \prod_{\gamma \in \mathcal{B}} \chi_+^{\lambda_\gamma - 1}(x_\gamma) \psi_G(x)^{\omega_G - \frac{D}{2}} \\ &\quad \times (1 - e^{-2\pi i \omega_G}) \chi_+^{-\omega_G}(\Phi_G(s, x)) |\Omega|_{X_{\mathcal{B}}},\end{aligned}$$

where we have used the boundary value representation,

$$\chi_+^{-\omega_G}(x) = \frac{1}{1 - e^{-2\pi i \omega_G}} ((x + i0)^{-\omega_G} - e^{-2\pi i \omega_G} (x - i0)^{-\omega_G}),$$

which follows as in Exam. 5.5.4. □

Remark 7.6.4. Suppose G is a massive, scalar, overall convergent graph in dimension $D^0 \in \mathbb{N}$, i.e. $\lambda_i^0 = 1$ for all $i \in G$, and $\omega_\gamma = |\gamma| - \frac{D}{2} h^1(G) > 0$ for all $\gamma \subseteq G$. We have the boundary value representation

$$\delta^{(\omega_G - 1)}(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\omega_G - 1} \delta(x) = \frac{1}{-2\pi i} \left((x + i0)^{-\omega_G} - (x - i0)^{-\omega_G}\right),$$

which follows directly from Exam. 5.5.6. An analogous argument gives then

$$\operatorname{Disc} I_G^{par}(\lambda^0, D^0, s) = -2\pi i \int_{X_{\mathcal{B}}(\mathbb{R})} \left(\delta^{(\omega_G - 1)}(\Phi_G(s, x))\right) \psi_G(x)^{\omega_G - \frac{D}{2}} |\Omega|_{X_{\mathcal{B}}}.$$

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